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AUSTRALIA

ENM1600

Engineering mathematics

Engineering mathematics refreshment

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Pre-test

Introduction

The pre-test includes questions from mathematics topics which are essential for your future progression in algebra and calculus.

How to do the written pre-test

The pre-test questions are included in this package of materials. Make sure when you do the questions you write out all your working. You should use a non-programmable calculator only for the calculations.

Note

Have you ever looked at a question and said, ‘Yes I can do that’ but when it comes down to it have trouble getting a correct answer? We have found that this often occurs and that students can fail as a consequence. We urge you not to just look at a question and think you can do it but to attempt and try to complete each question on the pre-test. You must to do this in order to get an accurate diagnosis of which areas you need to refresh.

Questions

Question 1

a. Place these numbers in order from the smallest to the largest

$3, -2, 1.5, -2\frac{1}{4}, \sqrt{5}, -\sqrt{2}$

b. Give three values of p which satisfy the inequality

$p \geq -5$

c. Complete $24 \div \square = -6$

d. Evaluate $-3(2 - 8)$

e. Estimate $56 + 23 \times 9246 \div 125$ by using appropriate rounding

f. Evaluate $\left\{20 - 3\left[3 + \left(\frac{12}{4}\right)^2\right]\right\}^3$

Question 2

a. Express $\frac{5}{6}$ as a fraction equivalent with a denominator of 24

b. Express $\frac{1}{8}$ as a decimal

c. Find 8% of 330 mL

d. Simplify the ratio 24:14

e. Evaluate $2\frac{1}{2} \times 4\frac{1}{3}$ and express your answer as a fraction

f. Evaluate $\frac{1}{4} \div \frac{5}{6} + \frac{3}{4} - \frac{5}{2} \times \frac{4}{3}$ and express your answer as a fraction

Question 3

a. Evaluate $27^{\frac{2}{3}}$

b. Simplify $(-2)^5 \times (-2)^{-8}$

c. Evaluate $(2.3 \times 10^2 - 4.4 \times 10^3)^2$

d. Express $\dfrac{16(a^2b^4)^{-\frac{1}{2}}}{b^{-3}}$ as a simple fraction involving no negative powers

e. Express 0.000 002 6 km in mm

Question 4

a. Factorize $6x^2 + x - 12$

b. Expand $(x + 1)(-2x + 1)(x - 3)$

c. Write this expression as a single fraction with no common factors $\dfrac{1}{x-3} - \dfrac{4}{x-2}$

d. Make t the subject of the equation $y = (8t + 3)^3 + 4$

e. Solve the quadratic equation for x, $3x^2 + 4x - 8 = 0$

f. Solve the cubic equation for x, $x^3 - 4x^2 + x + 6 = 0$

g. Solve for x, $|3x + 3| < 6$

h. By completing the square, find the values of a and b where $x^2 + 3x + 1 = (x + a)^2 - b^2$

i. Solve the following set of simultaneous equations

$x + y + z = 0$

$x - 3y + 2z = 1$

$2x - 2y + z = -1$

Question 5

a. If $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-1}$ calculate

i. $f(-1)$

ii. $f(x + h)$

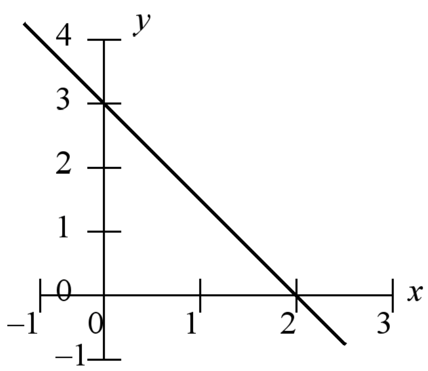
iii. $f(g(x))$

iv. What are the domain and range of $g(x)$

b. Write an equation for a straight line with slope of –4 and y-intercept of –3

c. Find the equation of the straight line passing through the points (–3,1) and (–1, –2)

d. Write an equation for the straight line below.



e. Sketch the graph of $y = -\dfrac{x}{2} + 2$

f.

i. Draw the graph of $y = x^2 + 7x + 6$

ii. Use the graph you have drawn to predict the y value when $x = –2.5$

g. What is the turning point of the function drawn in question 5 f. i.

h. Determine the centre and radius of the circle $x^2 + y^2 – 2x + 3y = 25$

i. Sketch the graph of $y = \dfrac{1}{x - 2}$

j. Indicate by a labelled sketch how you would graphically approximate the solution to the equation $x^2 - 1 = \sqrt{x}$

Question 6

a. Sketch a graph of $y = e^x$ and $y = \log_e x$

b. Make x the subject of the equation $y = 3e^x + 2$

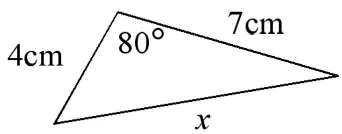
c. Evaluate using the logarithmic rules (do not use your calculator) $\log_2 4 – \log_2 2 + \log_2 1$

Question 7

a. Convert 329° to radians

b. Find all the angles between 0 and 2π radians that satisfy the equation $\sin A = 0.4$

c. In the triangle below find x



d. On the same set of axes sketch and label the graphs of

$y = \sin x$ and $y = \cos x$ for $–2\pi \le x \le 2\pi$

e. Complete the following statements

i. $\square + \cos^2\theta = 1$

ii. $1 + \square = \sec^2\theta$

iii. $\sin 2\theta = \square$

f. A surveyor attempting to find the height of a vertical cliff makes the following observations:

The angle of elevation from the ground to the top of the cliff is 30° at a certain distance away from the bottom of the cliff. But, the angle of elevation is 45° when 20m closer to the cliff.

What is the height of the cliff?

Module 1 – Fundamentals of arithmetic

Objectives

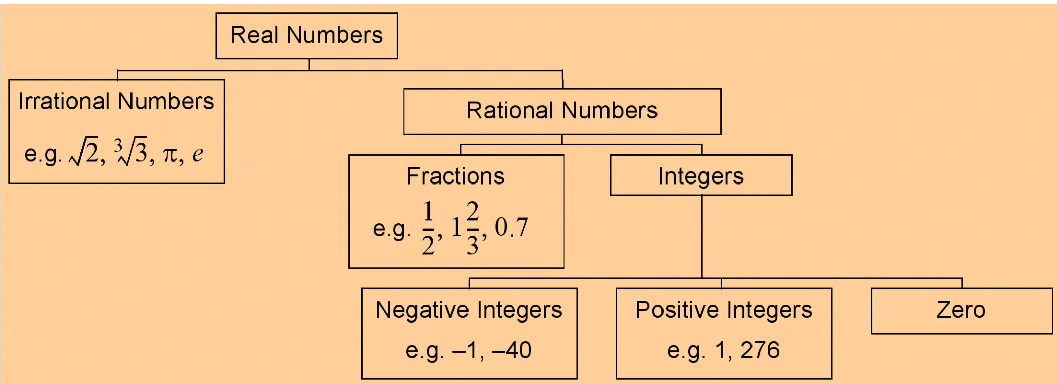
In this module you are required to be able to:

- apply the order of operations (by hand and with calculator) to calculations with all real numbers, including fractions
- demonstrate an understanding of index laws and apply these to simplifying problems involving: algebraic symbols, decimal numbers and numbers in scientific notation
- demonstrate an understanding of and apply scientific notation
- demonstrate an understanding of metric prefixes and convert between different metric units.

1.1 Review of the real number system

Throughout your studies you will be in contact with numbers of many types. Initially you will meet numbers from the real number system. These numbers form the basis of the mathematics that you will encounter in the following materials, although they are by no means the only types of numbers that exist.

To get an overall picture of which numbers are real numbers examine the following diagram:



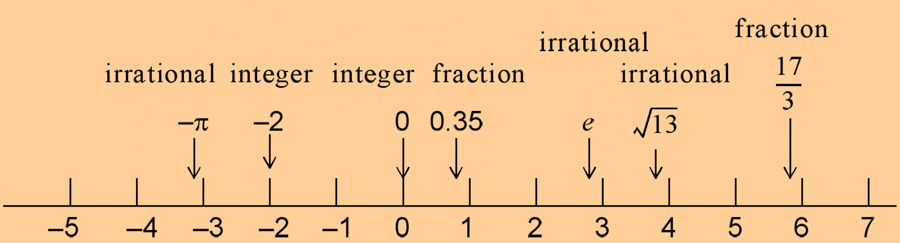
As you can see, the set of real numbers contains many subsets of other numbers. These can all be represented on the real number line.

Note the numbers π and e are special types of irrational numbers. You have probably heard of π and the number e is used in exponential expressions and is approximated by 2.71828. You will come across this in more detail in module 4.

Example

Identify the following types of real numbers and place them on the real number line:

$$-2,\ e,\ 0.35,\ \sqrt{13},\ \frac{17}{3},\ -\pi,\ 0.$$



1.2 Order of operations

To perform operations on these numbers mathematicians have established an accepted method for performing a series of calculations on groups of numbers. This is termed the order convention and specifies the order in which operations are to be performed.

It can be stated as:

Working from left to right evaluate:

Step 1 any expression in brackets
Step 2 any multiplications and divisions
Step 3 any additions and subtractions.

Note that it is extremely important that you can understand and use this order convention, for not only will it give you correct answers when doing arithmetic but it is the foundation upon which all further mathematics, in particular algebra, is based.

When undertaking any calculation it is always important to have an estimate of the answer as a check of your calculations. This is especially important when completing calculations using a calculator.

Examples

Example

Evaluate $3 + 2 \times 7$ (without using a calculator)

$3 + 2 \times 7$ perform multiplication

$= 3 + 14$ perform addition

$= 17$

Example

Evaluate $25 - (4 + 3 \times 2) \times 2$ (without using a calculator)

$= 25 - (4 + 6) \times 2$ perform multiplication inside brackets

$= 25 - 10 \times 2$ perform addition inside brackets

$= 25 - 20$ perform multiplication

$= 5$ perform subtraction

Example

Evaluate $221 - 3 \times (3 + 12 \div (3 + 1) - 1)$

$= 221 - 3 \times (3 + 12 \div 4 - 1)$ perform addition inside innermost brackets

$= 221 - 3 \times (3 + 3 - 1)$ perform division inside brackets

$= 221 - 3 \times 5$ perform addition and subtraction inside brackets

$= 221 - 15$ perform multiplication

$= 206$ perform subtraction

Example

Evaluate $1051 - 6 \times (25\ (3 + 10) \div 271 - 4)$

On calculator (Casio scientific):

1 0 5 1 – 6 × [(... 2 5 × [(... 3 + 1 0 ...)] ÷ 2 7 1 – 4 ...)] =

$= 1067.8044$

Exact answer is 1067.8044.

Estimation:

$\approx 1000 - 6 \times (30\ (3 + 10) \div 300 - 4)$ Estimation used to check the accuracy of the calculated answer

$\approx 1000 - 6 \times (30 \times 10 \div 300 - 4)$

$\approx 1000 - 6 \times (1 - 4)$

$\approx 1000 - 6 \times -3$

$\approx 1000 + 18$

≈ 1018

Yes, 1067.8044 is a reasonable answer.

Exercise 1.1

Evaluate the following **without** a calculator.

Estimate the final results before commencing.

a. $765 \div 15 + 822$

b. $89 + 21 - 48 \times 23$

c. $591 \times 376 + 523$

d. $895\,(622 + 479)$

e. $4763 + 395 \div 5 \times 16$

f. $52.1 - 3 \div 4.8 \times [4.1 - (2.1 + 7.6 \div 2)] + 4$

g. $4.8 \div -4 - [3\,(10 - 10 \div -2)]$

Solutions

Verify solutions with your calculator.

a. $765 \div 15 + 822$
$= 51 + 822$
$= 873$

Step 1

estimate: $765 \approx 800$
$15 \approx 20$
$800 \div 20 = \underline{40}$

calculate:

$$\begin{array}{r} \underline{51} \\ 15\overline{)765} \\ \underline{75} \\ 15 \\ \underline{15} \\ 0 \end{array}$$

check: yes, 51 is a reasonable answer

Step 2

estimate: $51 \approx 50$
$822 \approx 800$
$50 + 800 = \underline{850}$

calculate:

$$\begin{array}{r} 51 \\ +\underline{822} \\ \underline{873} \end{array}$$

check: yes, 873 is a reasonable answer

b. 89 + 21 – 48 × 23
= 89 + 21 – 1104
= 110 – 1104
= –994

Step 1

estimate: $48 \approx 50$
$23 \approx 20$
$50 \times 20 = \underline{1000}$

calculate:

$$\begin{array}{r} 48 \\ \underline{23} \\ 144 \\ \underline{960} \\ 1104 \end{array}$$

check: yes, 1104 is a reasonable answer

Step 2

estimate: $89 \approx 90$
$21 \approx 20$
$90 + 20 = \underline{110}$

calculate:

$$\begin{array}{r} 89 \\ +\underline{\ 21} \\ \underline{110} \end{array}$$

check: yes, 110 is a reasonable answer

Step 3

estimate: $110 \approx 100$
$1104 \approx 1000$
$100 - 1000 = \underline{-900}$

calculate:

$$\begin{array}{r} 1104 \\ -\underline{\ 110} \\ 994 \end{array}$$

$\therefore 110 - 1104 = -994$

check: yes, –994 is a reasonable answer

c. 591 × 376 + 523
= 222 216 + 523
= 222 739

Step 1

estimate: $591 \approx 600$
$376 \approx 400$
$600 \times 400 = \underline{240\ 000}$

calculate:

$$\begin{array}{r} 591 \\ \times\underline{\ 376} \\ 3546 \\ 41370 \\ \underline{177300} \\ 222216 \end{array}$$

check: yes, 222 216 is a reasonable answer

Step 2

estimate: $222\ 216 \approx 200\ 000$
$523 \approx 500$
$200\ 000 + 500 = \underline{200\ 500}$

calculate:

$$\begin{array}{r} 222216 \\ +\underline{\quad 523} \\ 222739 \end{array}$$

check: yes, 222 739 is a reasonable answer

d. $895\ (622 + 479)$
$= 895 \times 1101$
$= 985\ 395$

Step 1

estimate: $622 \approx 600$
$479 \approx 500$
$600 + 500 = \underline{1100}$

calculate:

$$\begin{array}{r} 622 \\ +\underline{\ 479} \\ 1101 \end{array}$$

check: yes, 1101 is a reasonable answer

Step 2

estimate: $895 \approx 900$
$1101 \approx 1000$
$900 \times 1000 = \underline{900\ 000}$

calculate:

$$\begin{array}{r} 895 \\ \times\underline{\ 1101} \\ 895 \\ 0000 \\ 89500 \\ \underline{895000} \\ 985395 \end{array}$$

check: yes, 985 395 is a reasonable answer

e. $4763 + 395 \div 5 \times 16$
$= 4763 + 79 \times 16$
$= 4763 + 1264$
$= 6027$

Step 1

estimate: $395 \approx 400$
$400 \div 5 = \underline{80}$

calculate:

$$\begin{array}{r} \underline{\quad 79} \\ 5\overline{)395} \\ 3\underline{5} \\ 45 \\ 4\underline{5} \\ 0 \end{array}$$

check: yes, 79 is a reasonable answer

Step 2

estimate: $79 \approx 80$
$16 \approx 20$
$80 \times 20 = \underline{1600}$

calculate:

$$\begin{array}{r} 79 \\ \times\underline{\ 16} \\ 474 \\ \underline{790} \\ \underline{1264} \end{array}$$

check: yes, 1254 is a reasonable answer

Step 3

estimate: $4763 \approx 5000$
$1264 \approx 1000$
$5000 - 1000 = \underline{6000}$

calculate:

$$\begin{array}{r} 4763 \\ +\ 1264 \\ 6027 \end{array}$$

check: yes, 6027 is a reasonable answer

f. 52.1 – 3 ÷ 4.8 × [4.1 – (2.1 × 7.6 ÷ 2)] + 4

5 2 • 1 – 3 ÷ 4 • 8 × [(... 4 • 1 – [(... 2 • 1 × 7 • 6 ÷ 2 ...)] ...)] + 4 =

= 58.525

estimate:

50 – 3 ÷ 5 × [4 – (2 × 8 ÷ 2)] + 4

≈ 50 – 3 ÷ 5[4 – 8] + 4

≈ 50 – 3 ÷ 5[–4] + 4

$\approx 50 - \frac{3}{5} \times -4 + 4; \quad \frac{3}{5} \times -4 \approx 2$

≈ 50 + 2 + 4

≈ 56

check: yes, 58.525 is a reasonable answer

g. 4.8 ÷ –4 – [3 (10 – 10 ÷ –2)]

4 • 8 ÷ 4 +/- – [(... 3 × [(... 1 0 – 1 0 ÷ 2 +/- ...)] ...)] =

= – 46.2

estimate:

5 ÷ –4 – [3 (10 – 10 ÷ –2)]

≈ 5 ÷ – 4 – [3 × 15]

$\approx \frac{5}{-4} - 45$

≈ –1–45

≈ –46

check: yes, –46.2 is a reasonable answer

1.3 Numerical fractions

There are many real life situations where whole numbers alone are not sufficient. Fractions of whole numbers are represented as

$$\frac{3}{4}$$

3 ← numerator

4 ← denominator

Fractions can be written in a number of forms.

Consider $\frac{3}{4}$, the numerator is less than the denominator. This is called a **proper fraction**.

Consider $\frac{12}{7}$, the numerator is greater than the denominator. This is called an **improper fraction**.

Consider $4\frac{2}{5}$, the whole number appears with a proper fraction. This is called a **mixed number**.

For calculations to be performed with fractions it is preferred that fractions be written in either proper or improper form.

Note that although fractional calculations may be performed on a calculator it is important that you are able to do these calculations from first principles. The operations will be used later in algebra and calculus.

1.3.1 Addition and subtraction of fractions

Consider $\frac{3}{10}+\frac{4}{10}=\frac{7}{10}$

These two fractions can be easily added because they are the same type of fraction (tenths), i.e. they have the same denominator.

Consider $\frac{1}{4}+\frac{2}{3}$

These two fractions cannot be easily added because they are different types of fractions (quarters and thirds).

To convert them to the same type of fraction, they have to be written as **equivalent fractions** with the same denominator.

$$\frac{1}{4} \times \frac{3}{3} = \frac{3}{12} \qquad (\frac{1}{4} \text{ and } \frac{3}{12} \text{ are equivalent fractions})$$

$$\frac{2}{3} \times \frac{4}{4} = \frac{8}{12}$$

Note: 12 is called the lowest common denominator for these two fractions.

To calculate

$$\frac{1}{4} + \frac{2}{3}$$

$$= \left(\frac{1}{4} \times \frac{3}{3}\right) + \left(\frac{2}{3} \times \frac{4}{4}\right)$$

$$= \frac{3}{12} + \frac{8}{12}$$

$$= \frac{11}{12}$$

Consider $3\frac{2}{5} + 2\frac{3}{8}$, in this calculation the fractions should first be converted to improper fractions then continue as before.

$$3\frac{2}{5} + 2\frac{3}{8}$$

$$= \frac{17}{5} + \frac{19}{8}$$

$$= \left(\frac{17}{5} \times \frac{8}{8}\right) + \left(\frac{19}{8} \times \frac{5}{5}\right)$$

$$= \frac{136}{40} + \frac{95}{40}$$

$$= \frac{231}{40}$$

$$= 5\frac{31}{40}$$

Example

Simplify $\frac{1}{6} - \frac{7}{9}$

The lowest common denominator is 18.

$$\frac{1}{6} - \frac{7}{9} = \left(\frac{1 \times 3}{6 \times 3}\right) - \left(\frac{7 \times 2}{9 \times 2}\right)$$

$$= \frac{3 - 14}{18}$$

$$= \frac{-11}{18}$$

Multiply $\frac{1}{6}$ by $\frac{3}{3}$ and $\frac{7}{9}$ by $\frac{2}{2}$ to make suitable equivalent fractions. Subtract as before.

Exercise 1.2

Evaluate:

a. $\frac{3}{7}+\frac{1}{2}$

b. $\frac{2}{3}-\frac{3}{4}$

c. $\frac{2}{7}+\frac{3}{14}$

d. $\frac{13}{15}-\frac{2}{5}$

e. $\frac{5}{6}+\frac{3}{4}$

f. $\frac{-5}{6}+\frac{9}{10}$

g. $\frac{5}{30}-\frac{7}{10}$

h. $\frac{3}{20}-\frac{1}{4}$

i. $\frac{3}{25}+\frac{57}{100}$

j. $\frac{5}{9}-\frac{5}{6}+\frac{1}{2}$

Solutions

a. $\frac{3}{7}+\frac{1}{2}$

$$= \frac{3\times 2}{7\times 2}+\frac{1\times 7}{2\times 7}$$

$$= \frac{6+7}{14}$$

$$= \frac{13}{14}$$

b. $\frac{2}{3}-\frac{3}{4}$

$$= \frac{2\times 4}{3\times 4}-\frac{3\times 3}{4\times 3}$$

$$= \frac{8-9}{12}$$

$$= \frac{-1}{12}$$

c. $\frac{2}{7}+\frac{3}{14}$

$$= \frac{2\times 2}{7\times 2}+\frac{3}{14}$$

$$= \frac{4+3}{14}$$

$$= \frac{7}{14}$$

$$= \frac{1}{2}$$

d. $\frac{13}{15}-\frac{2}{5}$

$$= \frac{13}{15}-\frac{2\times 3}{5\times 3}$$

$$= \frac{13-6}{15}$$

$$= \frac{7}{15}$$

e. $\frac{5}{6}+\frac{3}{4}$

$$=\frac{5\times 2}{6\times 2}+\frac{3\times 3}{4\times 3}$$
$$=\frac{10+9}{12}$$
$$=\frac{19}{12}$$
$$=1\frac{7}{12}$$

f. $\frac{-5}{6}+\frac{9}{10}$

$$=\frac{-5\times 5}{6\times 5}+\frac{9\times 3}{10\times 3}$$
$$=\frac{-25+27}{30}$$
$$=\frac{\cancel{2}^{\,1}}{\cancel{30}_{\,15}}$$
$$=\frac{1}{15}$$

g. $\frac{5}{30}-\frac{7}{10}$

$$=\frac{5}{30}-\frac{7\times 3}{10\times 3}$$
$$=\frac{5-21}{30}$$
$$=\frac{-\cancel{16}^{\,-8}}{\cancel{30}_{\,15}}$$
$$=\frac{-8}{15}$$

h. $\frac{3}{20}-\frac{1}{4}$

$$=\frac{3}{20}-\frac{1\times 5}{4\times 5}$$
$$=\frac{3-5}{30}$$
$$=\frac{-\cancel{2}^{\,-1}}{\cancel{20}_{\,10}}$$
$$=\frac{-1}{10}$$

i. $\frac{2}{25}+\frac{57}{100}$

$$=\frac{3\times 4}{25\times 4}+\frac{57}{100}$$
$$=\frac{12+57}{100}$$
$$=\frac{69}{100}$$

j. $\frac{5}{9}-\frac{5}{6}+\frac{1}{2}$

$$=\frac{5\times 2}{9\times 2}-\frac{5\times 3}{6\times 3}+\frac{1\times 9}{2\times 9}$$
$$=\frac{10-15+9}{18}$$
$$=\frac{\cancel{4}^{\,2}}{\cancel{18}_{\,9}}$$
$$=\frac{2}{9}$$

Examples

Example

Evaluate $-5\frac{1}{4}+3\frac{5}{6}$

$-5\frac{1}{4}+3\frac{5}{6}$ convert to improper fractions

$=\frac{-21}{4}+\frac{23}{6}$

$=\frac{-21\times 3}{4\times 3}+\frac{23\times 2}{6\times 2}$

$\frac{-63+46}{12}$ add as before

$=\frac{-17}{12}$

$=-1\frac{5}{12}$ convert to a mixed number

Example

Evaluate $5\frac{2}{3}-3\frac{1}{2}$

$5\frac{2}{3}-3\frac{1}{2}$ convert to improper fractions

$=\frac{17}{3}-\frac{7}{2}$

$=\frac{17\times 2}{3\times 2}-\frac{7\times 3}{2\times 3}$ subtract as before

$=\frac{34-21}{6}$

$=\frac{13}{6}$

$=2\frac{1}{6}$ convert to a mixed number

Exercise 1.3

Evaluate:

a. $3\frac{1}{4} + 2\frac{1}{2}$

b. $4\frac{3}{8} - 2\frac{5}{6}$

c. $3\frac{2}{3} + 1\frac{3}{4}$

d. $2\frac{5}{7} - 3\frac{2}{7}$

e. $5\frac{3}{10} - 4\frac{2}{5}$

f. $-2\frac{1}{2} + 5\frac{3}{5}$

g. $-3\frac{7}{9} + 2\frac{1}{6}$

h. $17\frac{1}{6} + 25\frac{1}{4}$

i. $50\frac{1}{2} - 45\frac{3}{4}$

j. $3\frac{2}{5} + 2\frac{1}{2} - 4\frac{3}{4}$

Solutions

a.
$$
\begin{aligned}
&3\frac{1}{4} + 2\frac{1}{2}\\
&= \frac{13}{4} + \frac{5}{2}\\
&= \frac{13}{4} + \frac{5 \times 2}{2 \times 2}\\
&= \frac{13 + 10}{4}\\
&= \frac{23}{4}\\
&= 5\frac{3}{4}
\end{aligned}
$$

b.
$$
\begin{aligned}
&4\frac{3}{8} - 2\frac{5}{6}\\
&= \frac{35}{8} - \frac{17}{6}\\
&= \frac{35 \times 3}{8 \times 3} - \frac{17 \times 4}{6 \times 4}\\
&= \frac{105 - 68}{24}\\
&= \frac{37}{24}\\
&= 1\frac{13}{24}
\end{aligned}
$$

c.
$$3\frac{2}{3}+1\frac{3}{4}$$
$$=\frac{11}{3}+\frac{7}{4}$$
$$=\frac{11\times 4}{3\times 4}+\frac{7\times 3}{4\times 3}$$
$$=\frac{44+21}{12}$$
$$=\frac{65}{12}$$
$$=5\frac{5}{12}$$

d.
$$2\frac{5}{7}-3\frac{2}{7}$$
$$=\frac{19}{7}-\frac{23}{7}$$
$$=\frac{19-23}{7}$$
$$=\frac{-4}{7}$$

e.
$$5\frac{3}{10}-4\frac{2}{5}$$
$$=\frac{53}{10}-\frac{22}{5}$$
$$=\frac{53}{10}-\frac{22\times 2}{5\times 2}$$
$$=\frac{53-44}{10}$$
$$=\frac{9}{10}$$

f.
$$-2\frac{1}{2}+5\frac{3}{5}$$
$$=\frac{-5}{2}+\frac{28}{5}$$
$$=\frac{-5\times 5}{2\times 5}+\frac{28\times 2}{5\times 2}$$
$$=\frac{-25+56}{10}$$
$$=\frac{31}{10}$$
$$=3\frac{1}{10}$$

g.
$$-3\frac{7}{9}+2\frac{1}{6}$$
$$=\frac{-34}{9}+\frac{13}{6}$$
$$=\frac{-34\times 2}{9\times 2}+\frac{13\times 3}{6\times 3}$$
$$=\frac{-68+39}{18}$$
$$=\frac{-29}{18}$$
$$=-1\frac{11}{18}$$

h.
$$17\frac{1}{6}+25\frac{1}{4}$$
$$=\frac{103}{6}+\frac{101}{4}$$
$$=\frac{103\times 2}{6\times 2}+\frac{101\times 3}{4\times 3}$$
$$=\frac{206+303}{12}$$
$$=\frac{509}{12}$$
$$=42\frac{5}{12}$$

i. $50\frac{1}{2} - 45\frac{3}{4}$

$= \frac{101}{2} - \frac{183}{4}$

$= \frac{101 \times 2}{2 \times 2} - \frac{183}{4}$

$= \frac{19}{4}$

$= 4\frac{3}{4}$

j. $3\frac{2}{5} + 2\frac{1}{2} - 4\frac{3}{4}$

$= \frac{17}{5} + \frac{5}{2} - \frac{19}{4}$

$= \frac{17 \times 4}{5 \times 4} + \frac{5 \times 10}{2 \times 10} - \frac{19 \times 5}{4 \times 5}$

$= \frac{68 + 50 - 95}{20}$

$= \frac{23}{20}$

$= 1\frac{3}{20}$

1.3.2 Multiplication of fractions

Examples

Example

Evaluate $\frac{3}{4} \times \frac{5}{8}$

$\frac{3}{4} \times \frac{5}{8}$

$= \frac{3 \times 5}{4 \times 8}$ multiply numerators
multiply denominators

$= \frac{15}{32}$

Example

Evaluate $\frac{7}{8} \times \frac{16}{21}$

$\frac{7}{8} \times \frac{16}{21}$

$= \frac{{}^{1}\cancel{7}}{{}_{1}\cancel{8}} \times \frac{\cancel{16}^{2}}{\cancel{21}_{3}}$ cancel common factors
(divide 16 and 8 by 8 and 7 and 21 by 7)

$= \frac{1 \times 2}{1 \times 3}$ multiply numerators
multiply denominators

$= \frac{2}{3}$

Example

Evaluate $-3\frac{1}{3} \times -7\frac{1}{2}$

$$-3\frac{1}{3} \times -7\frac{1}{2}$$

$$= \frac{-10}{3} \times \frac{-15}{2}$$

convert to improper fractions

$$= \frac{{}^{-5}\cancel{10} \times -\cancel{15}^{-5}}{{}_{1}\cancel{3} \times \cancel{2}_{1}}$$

cancel common factors
(divide –10 and 2 by 2 and –15 and 3 by 3)

$$= \frac{-5 \times -5}{1 \times 1}$$

multiply together

$$= \frac{25}{1}$$

$$= 25$$

write as a whole number

Exercise 1.4

Evaluate:

a. $\frac{5}{9} \times \frac{3}{20}$

b. $\frac{6}{9} \times \frac{18}{36}$

c. $\frac{3}{8} \times \frac{7}{9}$

d. $-\frac{4}{9} \times \frac{10}{15}$

e. $-\frac{7}{10} \times -\frac{51}{60}$

f. $2\frac{1}{2} \times \frac{4}{15}$

g. $5\frac{3}{10} \times 3\frac{2}{4}$

h. $-7\frac{3}{5} \times 4\frac{4}{19}$

i. $-3\frac{3}{4} \times -7\frac{1}{20}$

j. $-1\frac{1}{2} \times 1\frac{1}{9}$

Solutions

a. $\frac{5}{9} \times \frac{3}{20}$

$= {}^{1}\frac{\cancel{5}}{{}_{3}\cancel{9}} \times \frac{\cancel{3}^{1}}{\cancel{20}_{4}}$

$= \frac{1 \times 1}{3 \times 4}$

$= \frac{1}{12}$

b. $\frac{6}{9} \times \frac{18}{36}$

$= {}^{1}\frac{\cancel{6}}{{}_{1}\cancel{9}} \times \frac{\cancel{18}^{2}}{\cancel{36}_{6}}$

$= \frac{1 \times 2}{1 \times 6}$

$= \frac{1}{3}$

c. $\frac{3}{8} \times \frac{7}{9}$

$= {}^{1}\frac{\cancel{3}}{8} \times \frac{7}{\cancel{9}_{3}}$

$= \frac{1 \times 7}{8 \times 3}$

$= \frac{7}{24}$

d. $-\frac{4}{9} \times \frac{10}{15}$

$= -\frac{4}{9} \times \frac{\cancel{10}^{2}}{\cancel{15}_{3}}$

$= \frac{-4 \times 2}{9 \times 3}$

$= \frac{-8}{27}$

e. $-\frac{7}{10} \times -\frac{51}{60}$

$= \frac{-7}{10} \times \frac{-\cancel{51}^{-17}}{\cancel{60}_{20}}$

$= \frac{-7 \times -17}{10 \times 20}$

$= \frac{119}{200}$

f. $2\frac{1}{2} \times \frac{4}{15}$

$= \frac{5}{2} \times \frac{4}{15}$

$= \frac{{}^{1}\cancel{5} \times \cancel{4}^{2}}{{}_{1}\cancel{2} \times \cancel{15}_{3}}$

$= \frac{1 \times 2}{1 \times 3}$

$= \frac{2}{3}$

g. $5\frac{3}{10} \times 3\frac{2}{4}$

$= \frac{53}{10} \times \frac{14}{4}$

$= \frac{53 \times \cancel{14}^{7}}{{}_{5}\cancel{10} \times 4}$

$= \frac{371}{20}$

$= 18\frac{11}{20}$

h. $-7\frac{3}{5} \times 4\frac{4}{19}$

$= \frac{-38}{5} \times \frac{80}{19}$

$= \frac{{}^{-2}-\cancel{38} \times \cancel{80}^{16}}{{}_{1}\cancel{5} \times \cancel{19}_{1}}$

$= \frac{-2 \times 16}{1 \times 1}$

$= \frac{-32}{1}$

$= -32$

i.
$$-3\frac{3}{4}\times -7\frac{1}{20}$$
$$= \frac{-15}{4}\times\frac{-141}{20}$$
$$= \frac{-\cancel{15}^{\,-3}\times -141}{4\times \cancel{20}_{\,4}}$$
$$= \frac{-3\times -141}{4\times 4}$$
$$= \frac{423}{16}$$
$$= 26\frac{7}{16}$$

j.
$$-1\frac{1}{2}\times 1\frac{1}{9}$$
$$= \frac{-3}{2}\times\frac{10}{9}$$
$$= \frac{-\cancel{3}^{\,-1}\times \cancel{10}^{\,5}}{{}_{1}\cancel{2}\times \cancel{9}_{\,3}}$$
$$= \frac{-1\times 5}{1\times 3}$$
$$= -\frac{5}{3}$$
$$= -1\frac{2}{3}$$

1.3.3 Division of fractions

Dividing by a fraction is the same as multiplying by its reciprocal.

The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

The reciprocal of $\frac{1}{4}$ is $\frac{4}{1}$ (or 4).

Examples

Example

Evaluate $\frac{3}{4} \div \frac{2}{3}$

$\frac{3}{4} \div \frac{2}{3}$

$= \frac{3}{4} \times \frac{3}{2}$ multiply by the reciprocal of $\frac{2}{3}$ instead of dividing by it

$= \frac{3 \times 3}{4 \times 2}$

$= \frac{9}{8}$ as per multiplication

$= 1\frac{1}{8}$

Example

Evaluate $\frac{-15}{9} \div \frac{-25}{27}$

$\frac{-15}{9} \div \frac{-25}{27}$

$= \frac{-15}{9} \times \frac{-27}{25}$ multiply by the reciprocal of $-\frac{25}{17}$ instead of dividing by it

$= \frac{{}^{-3}\,-\cancel{15} \times -\cancel{27}\,{}^{-3}}{{}_{1}\,\cancel{9} \times \cancel{25}\,{}_{5}}$

$= \frac{-3 \times -3}{1 \times 5}$

$= \frac{9}{5}$ as per multiplication

$= 1\frac{4}{5}$

Example

Evaluate $5\frac{3}{4} \div 2\frac{1}{2}$

$$5\frac{3}{4} \div 2\frac{1}{2}$$

$$= \frac{23}{4} \div \frac{5}{2}$$

$$= \frac{23}{4} \times \frac{2}{5}$$ multiply by the reciprocal of $\frac{5}{2}$ instead of dividing by it

$$= \frac{23 \times \cancel{2}^{1}}{{}_{2}\cancel{4} \times 5}$$

$$= \frac{23 \times 1}{2 \times 5}$$ as per multiplication

$$= \frac{23}{10}$$

$$= 2\frac{3}{10}$$

Exercise 1.5

Evaluate:

a. $\frac{3}{4} \div \frac{9}{20}$

b. $\frac{25}{27} \div \frac{15}{18}$

c. $\frac{1}{2} \div \frac{3}{4}$

d. $\frac{-21}{25} \div \frac{35}{30}$

e. $2\frac{3}{5} \div 3\frac{1}{2}$

f. $4\frac{2}{5} \div 2\frac{7}{10}$

g. $-3\frac{9}{15} \div 1\frac{4}{5}$

h. $5\frac{1}{2} \div 3\frac{3}{4}$

i. $7\frac{5}{8} \div -3\frac{1}{2}$

j. $-2\frac{1}{4} \div -4\frac{3}{5}$

Solutions

a. $\frac{3}{4} \div \frac{9}{20}$

$= \frac{3}{4} \times \frac{20}{9}$

$= \frac{{}_{1}\cancel{3} \times \cancel{20}^{5}}{{}_{1}\cancel{4} \times \cancel{9}_{3}}$

$= \frac{1 \times 5}{1 \times 3}$

$= \frac{5}{3}$

$= 1\frac{2}{3}$

b. $\frac{25}{27} \div \frac{15}{18}$

$= \frac{25}{27} \times \frac{18}{15}$

$= \frac{{}^{5}\cancel{25} \times \cancel{18}^{2}}{{}_{3}\cancel{27} \times \cancel{15}_{3}}$

$= \frac{5 \times 2}{3 \times 3}$

$= \frac{10}{9}$

$= 1\frac{1}{9}$

c. $\frac{1}{2} \div \frac{3}{4}$

$= \frac{1}{2} \times \frac{4}{3}$

$= \frac{1 \times \cancel{4}^{2}}{{}_{1}\cancel{2} \times 3}$

$= \frac{1 \times 2}{1 \times 3}$

$= \frac{2}{3}$

d. $\frac{-21}{25} \div \frac{35}{30}$

$= \frac{-21}{25} \times \frac{30}{35}$

$= \frac{{}^{-3}-\cancel{21} \times \cancel{30}^{6}}{25 \times \cancel{35}_{1}}$

$= \frac{-3 \times 6}{25 \times 1}$

$= \frac{-18}{25}$

e. $2\frac{3}{5} \div 3\frac{1}{2}$

$= \frac{13}{5} \div \frac{7}{2}$

$= \frac{13}{5} \times \frac{2}{7}$

$= \frac{13 \times 2}{5 \times 7}$

$= \frac{26}{35}$

f. $4\frac{2}{5} \div 2\frac{7}{10}$

$= \frac{22}{5} \div \frac{27}{10}$

$= \frac{22}{5} \times \frac{10}{27}$

$= \frac{22 \times \cancel{10}^{2}}{{}_{1}\cancel{5} \times 27}$

$= \frac{22 \times 2}{1 \times 27}$

$= \frac{44}{27}$

$= 1\frac{17}{27}$

g.
$$
\begin{aligned}
& -3\frac{9}{15} \div 1\frac{4}{5} \\
&= \frac{-54}{15} \div \frac{9}{5} \\
&= \frac{-54}{15} \times \frac{5}{9} \\
&= \frac{{}^{-2}\cancel{-54} \times \cancel{5}^{1}}{{}_{1}\cancel{15} \times \cancel{9}_{1}} \\
&= \frac{-2 \times 1}{1 \times 1} \\
&= -2
\end{aligned}
$$

h.
$$
\begin{aligned}
& 5\frac{1}{2} \div 3\frac{3}{4} \\
&= \frac{11}{2} \div \frac{15}{4} \\
&= \frac{11}{2} \times \frac{4}{15} \\
&= \frac{11 \times \cancel{4}^{2}}{{}_{1}\cancel{2} \times 15} \\
&= \frac{11 \times 2}{1 \times 15} \\
&= \frac{22}{15} \\
&= 1\frac{7}{15}
\end{aligned}
$$

i.
$$
\begin{aligned}
& 7\frac{5}{8} \div -3\frac{1}{2} \\
&= \frac{61}{8} \div \frac{-7}{2} \\
&= \frac{61}{8} \times \frac{-2}{7} \\
&= \frac{61 \times -\cancel{2}^{-1}}{{}_{4}\cancel{8} \times 7} \\
&= \frac{61 \times -1}{4 \times 7} \\
&= \frac{-61}{28} \\
&= -2\frac{5}{28}
\end{aligned}
$$

j.
$$
\begin{aligned}
& -2\frac{1}{4} \div -4\frac{3}{5} \\
&= \frac{-9}{4} \div \frac{-23}{5} \\
&= \frac{-9}{4} \times \frac{-5}{23} \\
&= \frac{-9 \times -5}{4 \times 23} \\
&= \frac{45}{92}
\end{aligned}
$$

1.4 Indices

In order to be able to manipulate formulae and algebraic expressions, it is necessary to have a sound knowledge of powers (or indices) of numbers and how they may be combined. There is a set of laws controlling index operations which may be applied under almost any circumstances involving positive or negative numbers, whole numbers or fractions.

Powers using calculator

If m is a positive integer, then

$$a^m = a \times a \times a \ldots \text{ to } m \text{ factors}$$

In this form a is called the base and m the power (index or exponent).

Most scientific calculators have a 'Y^x' or 'X^y' key for raising a number to a power. Learn how to use this key.

Example

Use your calculator to evaluate

a. 5^4

b. 2.73^3

c. 0^6

d. $(-4)^3$

Solutions

a. 625

b. 20.346417

c. 0. Note that zero raised to any power is zero with the exception of 0^0 which cannot be determined.

d. Most calculators will not accept negative base numbers. You must calculate 4^3 and determine the sign yourself. The answer is – 64.

Multiplication

$a^m \times a^n = (a \times a \times \ldots \text{ to } m \text{ factors}) \times (a \times a \times \ldots \text{ to } n \text{ factors})$

$= a \times a \times \ldots \text{ to } (m + n) \text{ factors} = a^{m+n}$

Thus the × sign between a^m and a^n indicates that the indices are to be **added**,
e.g. $2^3 \times 2^4 = 2^7$. Note that this rule does **not** apply to numbers with different bases
e.g. $2^3 \times 3^2$ cannot be combined.

Division

$$\begin{aligned} a^m \div a^n &= \frac{a^m}{a^n} = \frac{a \times a \times \ldots \text{ to } m \text{ factors}}{a \times a \times \ldots \text{ to } n \text{ factors}} \\ &= a \times a \ldots \text{ to } (m - n) \text{ factors} \\ &= a^{m-n} \end{aligned}$$

Thus the ÷ sign between a^m and a^n indicates that the indices are to be **subtracted**
e.g. $\frac{3^9}{3^4} = 3^5$

Power of a power

$$\begin{aligned} \left(a^m\right)^n &= a^m \times a^m \ldots \text{ to } n \text{ factors} \\ &= a^{(m + m + \ldots \text{ to } n \text{ terms})} \\ &= a^{mn} \end{aligned}$$

Thus to raise a number with a power to another power, **multiply** the powers together
e.g. $(3^3)^2 = 3^6$

Power of a product

$$\begin{aligned} (ab)^m &= (ab) \times (ab) \times \ldots \text{ to } m \text{ factors} \\ &= (a \times a \times \ \ldots \ \text{ to } m \text{ factors}) \times (b \times b \times \ \ldots \ \text{ to } m \text{ factors}) \\ &= a^m \times b^m \end{aligned}$$

A product raised to a power is the product of each term raised to that power
e.g. $(3 \times 2)^2 = 3^2 \times 2^2 = 6^2 = 36$.

Fractional index

Consider the symbol $9^{\frac{1}{2}}$. It cannot be interpreted in the form $9^2 = 9 \times 9$.

If we use the multiplication law, we get

$$9^{\frac{1}{2}} \times 9^{\frac{1}{2}} = 9^{\frac{1}{2}+\frac{1}{2}} = 9^1 = 9$$

$$\therefore \left(9^{\frac{1}{2}}\right)^2 = 9$$

But $3^2 = 9$ therefore $9^{\frac{1}{2}} = 3 = \sqrt{9}$

Consider $a^{\frac{1}{3}}$

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a$$

$$\therefore \left(a^{\frac{1}{3}}\right)^3 = a$$

$$\therefore a^{\frac{1}{3}} = \sqrt[3]{a} \text{ (cube root of } a\text{)}$$

Similarly, $\left(a^{\frac{3}{4}}\right)^4 = a^3$

$$a^{\frac{3}{4}} = \sqrt[4]{a^3} \text{ (fourth root of } a^3\text{)}$$

In general, $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

For example, $9^{\frac{3}{2}} = \sqrt{9^3} = \sqrt{9^2 \times 9} = 9\sqrt{9} = 9 \times 3 = 27$

or, $9^{\frac{3}{2}} = \left(3^2\right)^{\frac{3}{2}} = 3^3$

Zero index

Consider the symbol 4^0 which cannot be interpreted in the form

$4^m = 4 \times 4 \times \ldots$to m factors.

However,

$4^0 \times 4^3 = 4^{0+3} = 4^3$

Dividing by 4^3 gives

$4^0 = 1$

In general, $a^0 = 1$

Thus, any number (except zero) raised to the power zero is 1. (0^0 is indeterminate).

Note that a^0 is not the same as a, which is really a^1.

Negative index

Consider 3^{-2} which cannot be interpreted in the form

$$3^2 = 3 \times 3$$

$$3^{-2} \times 3^2 = 3^{-2+2} = 3^0 = 1$$

Dividing by 3^2 gives

$$3^{-2} = \frac{1}{3^2}$$

In general, $a^{-m} = \dfrac{1}{a^m}$

Thus, a negative index indicates that the number is the reciprocal of the number with a positive index

e.g. $3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$

Note also that only the number is inverted and not its index.

e.g. $3^{-\frac{1}{2}}$ **is not** $\dfrac{1}{3^2}$ **or** $\dfrac{-1}{3^2}$ **but** $\dfrac{1}{3^{\frac{1}{2}}}$

Examples

Examples

Use index laws to simplify

a. $8^{\frac{2}{3}} \times 9^{\frac{3}{2}}$

b. $\dfrac{8a^{-2}}{a^3}$

c. $3^{-2.5} \times 3^{3.5}$

d. $\left(x^2 y^{-4}\right)^{\frac{1}{2}}$

e. $\dfrac{8\left(a^3 b^6\right)^{-\frac{1}{3}}}{b^{-5}}$

Solutions

a. $\left(2^3\right)^{\frac{2}{3}} \times \left(3^2\right)^{\frac{3}{2}} = 2^2 \times 3^3 = 4 \times 27 = 108$

b. $\dfrac{8a^{-2}}{a^3} = \dfrac{8}{a^2 \cdot a^3} = \dfrac{8}{a^5} = 8a^{-5}$

c. $3^{-2.5+3.5} = 3^1 = 3$

d. $\left(x^2\right)^{\frac{1}{2}} \times \left(y^{-4}\right)^{\frac{1}{2}} = x^1 \times y^{-2} = x^1 \times \dfrac{1}{y^2} = \dfrac{x}{y^2}$

e. $\dfrac{8\left(a^3\right)^{-\frac{1}{3}} \cdot \left(b^6\right)^{-\frac{1}{3}}}{b^{-5}} = 8a^{-1} \cdot b^{-2} \cdot b^5 = 8 \times \dfrac{1}{a} \times b^{-2+5} = \dfrac{8}{a} \times b^3 = \dfrac{8b^3}{a}$

Examples

Simplify

a. $\dfrac{5^{-4} \times 5^{\frac{7}{2}}}{125^{\frac{1}{3}}}$

b. $\dfrac{\left(m^{3} \times n^{-2}\right)^{2}}{mn^{-3}}$

c. $\left(-8\right)^{\frac{1}{3}} + \sqrt[4]{16}$

d. $27^{\frac{2}{3}} - \left(\dfrac{1}{3}\right)^{-3}$

e. $a^{4.2} - a^{3.1} + a^{0.9}$

f. $\sqrt{x^{2} + x^{4}}$

Solutions

a. $\dfrac{5^{-4} \times 5^{3.5}}{5^{1}} = \dfrac{5^{-0.5}}{5^{1}} = 5^{-0.5-1} = 5^{-1.5} \text{ or } \dfrac{1}{5\sqrt{5}}$

b. $\dfrac{\left(m^{3}\right)^{2} \times \left(n^{-2}\right)2}{mn^{-3}} = \dfrac{m^{6}}{m^{1}} \times \dfrac{n^{-4}}{n^{-3}} = m^{6-1} \times n^{-4+3} = m^{5} \times n^{-1} = m^{5} \times \dfrac{1}{n} = \dfrac{m^{5}}{n}$

c. $\left(-2^{3}\right)^{\frac{1}{3}} + \left(2^{4}\right)^{\frac{1}{4}} = -2^{1} + 2^{1} = 0$

d. $\left(3^{3}\right)^{\frac{2}{3}} - \left(3^{-1}\right)^{-3} = 3^{2} - 3^{3} = 9 - 27 = -18$

e. No simplification possible

f. $\sqrt{x^{2}\left(1 + x^{2}\right)} = \sqrt{x^{2}}\sqrt{1 + x^{2}} = x\sqrt{1 + x^{2}}$

Exercise 1.6

Reduce the following expressions to simplified forms without using your calculator.

a. $\dfrac{16^{\frac{1}{2}} \times 3^{\frac{3}{2}}}{2^3 \times 2^{-1}}$

b. $\dfrac{9^{\frac{3}{2}} \times 4^{\frac{5}{2}}}{3^2 \times 2^3}$

c. $\dfrac{\left(4^2\right)^3 \times 9^2}{64^2 \times 27^{\frac{4}{3}}}$

d. $\dfrac{7^3 \times 5^4}{35^2}$

e. $\dfrac{6^3 \times 2^2}{3^2 \times 2^3}$

f. $\dfrac{121^{\frac{5}{2}} \times 49^{\frac{3}{2}}}{11^3 \times 7^2}$

g. $\dfrac{8a^2b^3}{(ab)^4}$

h. $\dfrac{\sqrt{16a^2b}}{a^3b^{\frac{3}{2}}}$

i. $\dfrac{\sqrt[3]{27a^4b^2}}{\sqrt{81a^3b^4}}$

j. $\dfrac{\left(a^3b^{-3}\right)^3}{ab^{-2}}$

k. $(-27)^{\frac{1}{3}} + \sqrt[4]{81}$

l. $121^{\frac{3}{2}} - 64^{\frac{4}{3}}$

Solutions

a. $3^{\frac{3}{2}}$

b. 12

c. 1

d. 175

e. 12

f. 847

g. $\dfrac{8}{a^2b}$

h. $\dfrac{4}{a^2b}$

i. $\dfrac{1}{3a^{\frac{1}{6}}b^{\frac{4}{3}}}$

j. $\dfrac{a^8}{b^7}$

k. 0

l. 1075

1.5 Scientific notation

Consider the following measurements:

Breaking stress of steel = 430 000 000 Pascals

Wavelength of red light = 0.000 000 644 metres

Mass of virus = 0.000 000 000 000 000 0003 kilograms

These very large and very small numbers involving long strings of zeros are too difficult to handle as they stand, particularly using calculators. Such numbers are written using the 'scientific notation' which involves moving the decimal place to a convenient point (usually to get a number between 1 and 10) and writing the number of places moved as a power of 10.

For instance, the breaking stress of steel becomes 4.3×10^8 Pa since the decimal point was moved 8 places to the left. Similarly, the other two measurements become 6.44×10^{-7}m and 3×10^{-19} kg. **The negative index indicates that the decimal was moved to the right, i.e. the number is less than 1.**

An easy way to convert numbers is to remember that as the number gets smaller the index gets larger, and vice versa. Beware of negative indices, e.g. 10^{-7} **is larger** than 10^{-19}.

When calculating with scientific numbers, you must use the index laws to find the powers of 10.

Examples

Examples

Convert the following to the scientific notation.

a. 1072000

b. 0.00371

c. 2421.372

Solutions

a. 1.072×10^6

b. 3.71×10^{-3}

c. 2.421372×10^3

Examples

Change the decimal point to make the number fall in the range 1 to 10.

a. 741.3×10^{7}

b. 0.034×10^{-2}

c. 1820×10^{-5}

Solutions

a. 7.413×10^{9}

b. 3.4×10^{-4}

c. 1.820×10^{-2}

Examples

Simplify the following, using the scientific notation throughout.

a. $4.37 \times 10^{105} + 2.41 \times 10^{104}$

b. $(2.5 \times 10^{-3}) \times (4.4 \times 10^{14})$

c. $\dfrac{7.8 \times 10^{4}}{2 \times 10^{-3}}$

d. $\sqrt{1.6 \times 10^{-5}}$

Solutions

a. Recall that numbers with different indices cannot be added or subtracted. One must be converted to the same index as the other.

$$\begin{aligned} \therefore 4.37 \times 10^{105} + 2.41 \times 10^{104} &= 4.37 \times 10^{105} + 0.241 \times 10^{105} \\ &= (4.37 + 0.241) \times 10^{105} \\ &= 4.611 \times 10^{105} \end{aligned}$$

b.
$$\begin{aligned} (2.5 \times 4.4) \times (10^{-3} \times 10^{14}) &= 11 \times 10^{-3+14} \\ &= 11 \times 10^{11} \\ &= 1.1 \times 10^{12} \end{aligned}$$

c.
$$\begin{aligned} \frac{7.8 \times 10^{4}}{2 \times 10^{-3}} &= \frac{7.8}{2} \times 10^{4} \times 10^{3} \\ &= 3.9 \times 10^{4+3} \\ &= 3.9 \times 10^{7} \end{aligned}$$

Note that the index of –3 refers only to 10 and not to 2. Thus when 10^{-3} is brought to the top to become 10^{3}, the number 2 remains underneath.

d. $\sqrt{1.6 \times 10^{-5}} = \sqrt{1.6} \times \sqrt{10^{-5}} = 1.2649 \times 10^{-2.5}$

Although this is correct it is not a scientific number since the index should indicate the number of places the decimal point was moved. The index must be a **whole** number. Thus we must change the problem first, e.g.

$$\sqrt{1.6 \times 10^{-5}} = \sqrt{16 \times 10^{-6}} = \sqrt{16} \times \sqrt{10^{-6}} = 4 \times 10^{-3}$$

or

$$\sqrt{1.6 \times 10^{-5}} = \sqrt{0.16 \times 10^{-4}} = \sqrt{0.16} \times \sqrt{10^{-4}} = 0.4 \times 10^{-2}$$

$$= 4 \times 10^{-3}$$

Exercise 1.7

1. Convert the following numbers to scientific notation.

a. 221.6×10^6 b. 0.0421×10^{-5} c. 3124.8×10^{-4}

d. 16.185×10^4 e. 11.214×10^{-2} f. 1121.4

g. 3124.8×10^4 h. 0.0016185 i. 0.00045×10^6

j. 0.00045×10^{-6}

2. Simplify the following using scientific notation throughout.

a. $5.684 \times 10^3 + 0.218 \times 10^2$

b. $1.68 \times 10^3 + 0.64 \times 10^2$

c. $25.7824 \times 10^{60} + 327.987 \times 10^{59} - 3204.68 \times 10^{58}$

d. $\sqrt{1.44 \times 10^{-4}}$

e. $\sqrt[3]{2.7 \times 10^7}$

f. $\sqrt{1.68 \times 10^5 + 0.42 \times 10^6 + 4.12 \times 10^4}$

Solutions

1. a. 2.216×10^8 b. 4.21×10^{-7} c. 3.1248×10^{-1}

d. 1.6185×10^5 e. 1.1214×10^{-1} f. 1.1214×10^{-3}

g. 3.1248×10^7 h. 1.6185×10^{-3} i. 4.5×10^2

j. 4.5×10^{-10}

2. a. 5.7058×10^3 b. 1.744×10^3 c. 2.65343×10^{61}

d. 1.2×10^{-2} e. 3×10^2 f. 1×10^3

1.6 Metric units

The metric system uses a number of units for length, mass, time, etc. to which are attached various prefixes to indicate the size of the unit. The most common units are:

Measurement	Unit	Symbol
Length	metre	m
Mass	gram	g
Volume	litre	L
Time	second	s
Force	Newton	N
Power	Watt	W
Pressure	Pascal	Pa

The system of prefixes is based on powers of 10^3. Any prefix may be attached to any unit.

Prefix	Factor	Symbol
Giga	10^9	G
Mega	10^6	M
Kilo	10^3	k
base unit	$10^0 = 1$	
Milli	10^{-3}	m
Micro	10^{-6}	μ
Nano	10^{-9}	n
Pico	10^{-12}	p
*Centi	10^{-2}	c

* Centi is not properly part of the system as it is not a power of 10^3. However, it is commonly used and must be included.

Numbers should always be expressed in the range from 1 to 999. This is achieved by selecting the appropriate prefix, e.g. 2487mL becomes 2.487L and 0.047Mg becomes 47kg. When changing prefixes, you should remember that **if you make the number smaller**, **the prefix becomes larger and vice versa**. Since the prefixes are based on powers of 10^3, the decimal point moves in steps of three places.

Examples

Express the following in the appropriate prefix and unit using the proper symbols.

a. 0.0124 kilograms

b. 472×10^4 millimetres

c. 2472 kilopascals + 1.34 megapascals

d. 12000×120 microseconds

e. 1784 centimetres

Solutions

a. $0.0124 \text{ kg} = 0.0124 \times 10^3 \text{g} = 12.4 \text{ g}$

b. $472 \times 10^4 \text{ mm} = 4720000\text{mm} = 4720000 \times 10^{-3}\text{m}$

$= 4720\text{m} = 4720 \times 10^{-3}\text{km} = 4.72\text{km}$

c. 2472 kPa + 1.34 MPa = 2.472 MPa + 1.34 MPa = 3.812 MPa

d. 12000×120 μs = 1440000 μs = 1440 ms = 1.44 s

e. 1784 cm = 17.84 m (Note for 'centi' the decimal moves only 2 places)

Note again that when adjusting the decimal point in the scientific notation, if the number is made larger, then the index becomes smaller and vice versa. Take care with negative indices e.g. 10^{-4} is smaller than 10^{-3}.

Exercise 1.8

Express the following in the appropriate prefix and unit using the correct symbols.

a. 240800g	b. 16800m	c. 1910L	d. 0.678kg
e. 0.068m	f. 0.62L	g. 16425mg	h. 15×10^3mm
i. 26×10^6mL	j. 0.000625μg	k. 0.006nm	l. 0.0249kL

Solutions

a. 240.8kg	b. 16.8km	c. 1.91kL	d. 678g
e. 68mm	f. 620mL	g. 16.425g	h. 15m
i. 26kL	j. 625pg	k. 6pm	l. 24.9L

Module 1: Self assessment

Questions 1.1

1. Evaluate $23.1 - 2.1 \div (6.7 - 3 \times 4.1)$
2. Evaluate $7\frac{1}{8} - 8\frac{1}{9}$
3. Evaluate $4\frac{3}{5} \times 1\frac{1}{5}$
4. Evaluate 4^{-3}
5. Evaluate $27^{\frac{4}{3}}$
6. Express $\dfrac{\left(25a^4b^2\right)^{\frac{1}{2}}}{b^{-2}}$ as a single fraction involving no negative powers
7. Express 543.28 in scientific notation
8. Express the sum of 3 000 000 mg and 167 kg in Mg
9. Evaluate $(2.3 \times 10^2 - 4.1 \times 10^{-1})^2$

Questions 1.2

1. Evaluate $4.1 + 3.2 \times (-3.76 - 0.01 \div 4)$
2. Evaluate $2\frac{1}{7} - 6\frac{2}{5}$
3. Evaluate $-2\frac{1}{3} \times 4\frac{1}{7}$
4. Evaluate $216^{\frac{1}{3}}$
5. Evaluate $36^{\frac{3}{2}}$
6. Express $\frac{\left(9a^4b^2\right)^{\frac{1}{2}}}{b^{-4}}$ as a single fraction involving no negative powers
7. Express 0.000 077 1 in scientific notation
8. Evaluate $(-3.2 \times 10^{-1} + 2.1 \times 10^{2})^{3}$
9. Express the sum of 34.5 kL and 490 000 mL in litres

Module 2 – Basic algebra

Objectives

In this module you are required to be able to:

- apply order of operations to algebraic calculations
- factorise and expand algebraic expressions
- convert algebraic fractions to a single term
- manipulate algebraic equations and inequations in order to change the subject of the equation
- solve linear, quadratic and cubic equations
- express inequations in interval notation.

2.1 Variables and expressions

A **variable** is a term that is used to indicate that that symbol may take various values. Usually a letter of the alphabet is used to represent a variable. Sometimes we call these letters pronumerals because they act as a numeral. An expression is just a group of variables connected by arithmetic operations.

To the trained eye, variables and expressions exist in and can be used to help solve many of the problems with which a professional may be confronted. The skill of the person with mathematics training is that they can reduce such problems, in areas about which they may know nothing, into a generalised expression they can work with. Also, variables and operators allow us to make long word statements into brief mathematical expressions.

Example

A car's cooling system contains 10 litres of a mixture of water and antifreeze which is 25% antifreeze. How much of this mixture must be drained out and replaced with pure antifreeze so that the resulting 10 litres will be 40% antifreeze?

This can be summarised by the equation

$$\underbrace{\frac{40}{100}\times 10}_{\text{40\% of 10 litres}} = \underbrace{(10-x)}_{\text{original volume of mixture}}\frac{25}{100}+x$$

where x is the volume (in litres) to be drained off.

i.e.

$$(10-x)\frac{1}{4}+x = 4$$

When rewriting written expressions as mathematical expressions:

1. Carefully define any unknowns, stating units if they occur.
2. Translate the information in the problem into expressions using as few of the defined unknowns as possible.

   Once the above is completed then the mathematical expressions could be used to solve the problems.

   Consider the following simple situations.

   a. I was born three years before my sister; so I am always three years older than her. Knowing my age, I can calculate my sister's age by subtracting 3 from my age.

My age	My sister's age
20	17
12	9
43	40

   If we let m represent my age, then

   Sister's age = $m - 3$

   b. Ben is five times older than John; so if John is 10, Ben will be 50; if John is 3, Ben will be 15.

   If Ben is 30, then John will be one-fifth of the age, so John will be 6.

   If we let Ben's age be x then John's age will be $\frac{1}{5}x$.

   John's age = $\frac{1}{5}x$.

   c. A certain bank is offering, at present, an effective interest rate of 15% for fixed term deposits of 1 year for deposits over \$1 000. The interest is calculated by finding 15% of the amount in the deposit.

   The equation could be Interest = $0.15 \times A$, where A represents the amount deposited. This formula could also be written as Interest = $0.15A$.

   Since it doesn't matter which letters are chosen for variables in a formula (provided that it is understood what the letters mean), this formula could also be written as:

   Interest = $0.15\,x$ where x represents the amount deposited.

   It is most important that the **meaning** of variables is understood.

Now look at some examples where we find these equations:

Examples

Example

I am one third of my brother's age. Together our ages are 40. How old am I?

If I am x years old, then my brother is $3x$ years old. Together our ages are 40

so $x + 3x = 40$

Example

I invest some money in a bank offering 8.5% interest p.a. (per annum). Write a formula for finding the amount in the bank after one year.

If I invest \$2 500, how much money would I have after one year?

Let x be the amount invested

Amount $= x + 8.5\%$ of x

Amount $= x + 0.085 \times x$

Exercise 2.1

Consider the following word problems. Develop an equation for each question.

1. The sum of three consecutive positive integers is 15. Write an equation to find the integers. (Hint: Let the smallest integer be x, then the next integer is $x + 1$ and the largest integer is $x + 2$).

2. A piece of wire 63 cm long is to be cut into two pieces. One piece is to be 15 cm shorter than the other piece. Write an equation to find the length of the two pieces. (Hint: Let x be the length of the longer piece, then $x - 15$ will be the length of the shorter piece).

3. The length of a rectangle is 5 cm more than its width. If its perimeter is 44 cm write an equation to find the length and width of the rectangle.

4. The length of the base of a triangle is half the perpendicular height of the triangle. If the area of the triangle is 20.25 m^2, write an equation to find its base length and perpendicular height. (Hint: Let h be the perpendicular height, then the base length will be $\frac{h}{2}$).

5. An electricity board offers a choice between two tariffs.

   Tariff 1: a fixed charge of \$11 per quarter plus a charge of 2.5 cents per unit consumed.

   Tariff 2: a fixed charge of \$16 per quarter plus a charge of 2 cents per unit consumed.

   Write an equation to find how many units need to be consumed in a quarter for the overall cost under both tariffs to be the same? (Hint: Let n be the number of units that need to be consumed for overall costs to be equal. Find the cost under Tariff 1 for n units and the cost under Tariff 2 for n units and equate the costs).

Solutions

1. Let the smallest integer be x, then the next integer is $x + 1$ and the last integer is $x + 2$.

   Their sum is 15 means

   $x + x + 1 + x + 2 = 15$

   $x + x + 1 + x + 2 = 15$

   $x + x + x + 1 + 2 = 15$

   $x + x + x + 1 + 2 = 15$ Adding x's is the same as adding numbers

   $3x + 3 = 15$

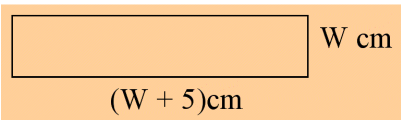
2. Let x be the length in cm of the longer piece of wire, then the shorter piece will be $(x – 15)$ cm in length.

   Joined together the two pieces make 63 cm. That is:

   $x + x – 15 = 63$

   $2x –15 = 63$

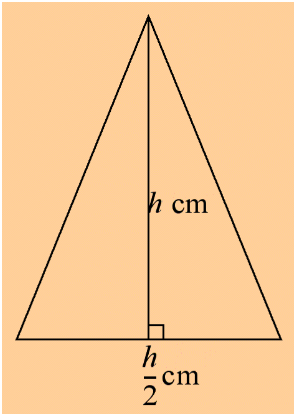
3. Let the width of the rectangle be W cm, then the length will be $(W + 5)$ cm.



   The perimeter of a rectangle = 2(length + width).

   So for this case $44 = 2((W + 5) + W)$

4. Let h be the perpendicular height of the triangle, then the base length is $\frac{h}{2}$



The area of a triangle = $\frac{1}{2}$ (base length × perpendicular height)

So for this case

$$20.25 = \frac{1}{2}\left(\frac{h}{2} \times h\right)$$

$$20.25 = \frac{1}{2} \times \frac{h^2}{2}$$

5. Let n be the number of units consumed that makes the total cost for each tariff equal.

Tariff 1

Total cost = Fixed cost + 2.5 cents per unit consumed

Total cost = 11 + 0.025n (changing 2.5 cents into 0.025 dollars) (in dollars)

Tariff 2

Total cost = Fixed cost + 2 cents per unit consumed

Total cost = 16 + 0.02n (in dollars)

If the total costs are to be equal,

$11 + 0.025n = 16 + 0.02n$

2.2 Algebraic expressions

Once mathematical expressions are developed then we need the skills to be able to manipulate these expressions or solve equations. The principles of order of operations which were developed in section 1.2 are used for such manipulations.

> The symbols +, –, ×, ÷ have exactly the same meaning in algebra as in arithmetic. Also the Order of Operations and the power rules we met earlier in arithmetic are the same in algebra.

2.2.1 Addition and subtraction of like terms

Like terms are those terms which contain the **same power of the same variable**.

> $6a$ and $2a$ are like terms because they have the same power of a (the power of a is 1). The coefficients do not influence whether terms are said to be like or not. $5x^3$ and $-7x^3$ are also like because each has the same power of x, that is, x^3.

Terms which are not like are called **unlike**. For example:

> $6x^4$ and $3y^4$ are unlike because they contain **different variables** (even though the same index is involved in both).
>
> $2x^2$ and $6x$ are unlike because they contain **different powers** (even though the variable is the same).
>
> $-3a$ and $-3z$ are unlike because they have **different variables** (remember, the coefficient has nothing to do with determining whether terms are like or unlike).

Example

Sort the following expressions into groups of like terms.

For example, with $7x^2$, $5x$, $3x^2$, $6x$ we would group $7x^2$ and $3x^2$ together and $5x$ and $6x$ together.

With the following expressions $3x$, $5a$, $7a^2$, $6x^2$, $7x$, $9a$, $17x$, $5x^2$, $6a^2$, $8x$, $9x^2$, $12a$, $-4x$, $-11x^2$, $-3a$, $-2x$, $2a^2$ we would group them as:

$3x$, $7x$, $17x$, $8x$, $-4x$, $-2x$

$5a$, $9a$, $12a$, $-3a$

$7a^2$, $6a^2$, $2a^2$

$6x^2$, $5x^2$, $9x^2$, $-11x^2$

We can now use the idea of like terms to simplify expressions.

Consider the expression $x + x$

Firstly determine if they are like terms. Yes, they are! (Because they are the same power of the same variable – both are x which means x^1.)

Now look at the coefficients. If there is no constant in front of the variable, then the coefficient is inferred to be 1.

If we have one x and add another x to it, we will have two x's.

Algebraically, $x + x = 2x$

Now look at the following examples

Examples

Example

Simplify $3x + 2x$

$3x + 2x$	Like terms? Yes – same power of the same variable
$= (3 + 2)x$	
$= 5x$	Simplify

Example

Simplify $5x^3 + 6x^3$

$5x^3 + 6x^3$	Like terms? Yes – same power of the same variable
$= (5 + 6)x^3$	
$= 11x^3$	Simplify

Example

Simplify $6x + 5x^2 - 2x - 4x^2$

$6x + 5x^2 - 2x - 4x^2$ Like terms? Yes – 2 pairs

$= (5x^2 - 4x^2) + (6x - 2x)$ Group like terms together. Begin with the highest powers and work down to the lowest

$= (5 - 4)x^2 + (6 - 2)x$

$= x^2 + 4x$ Simplify

Example

Simplify $-3x + 4y + xy + 8x - y$

$-3x + 4y + xy + 8x - y$ Like terms? Yes – 2 pairs and 1 other term

$= (8x - 3x) + (4y - y) + xy$ Group like terms together.

$= (8 - 3)x + (4 - 1)y + xy$

$= 5x + 3y + xy$ Simplify

Example

Simplify $3x^{\frac{1}{2}} + 5x^{-7} + 2x^{-7} - 9x^{\frac{1}{2}} - 3x^{-7} - 5x^{-7}$

$3x^{\frac{1}{2}} + 5x^{-7} + 2x^{-7} - 9x^{\frac{1}{2}} - 3x^{-7} - 5x^{-7}$ Like terms? Yes – 2 groups

$= \left(3x^{\frac{1}{2}} - 9x^{\frac{1}{2}}\right) + \left(5x^{-7} + 2x^{-7} - 3x^{-7} - 5x^{-7}\right)$

$= (3 - 9)\,x^{\frac{1}{2}} + (5 + 2 - 3 - 5)\,x^{-7}$

$= -6x^{\frac{1}{2}} - x^{-7}$ Simplify

Remember

1. You can only add or subtract like terms.
2. If a term is just x, x^2, etc., then the coefficient is one.
3. Take care when regrouping terms with negatives.

Exercise 2.2

a. $5x + 2x$

b. $-6x^2 - 5x^2$

c. $6.2a^3 + 3.7a^3$

d. $7a^{\frac{1}{2}} + 3a^{\frac{1}{2}} - 9a^{\frac{1}{2}}$

e. $-5x^4 - 3x^4 - 2.5x^4$

f. $5x + 2y^{-5} + 3x + 7y^{-5}$

g. $4x^{-2} + 2x + 6x^{-2} + 7x$

h. $7n^2 - 3n^{-\frac{1}{2}} + 2.6n^2 - 5.7n^{-\frac{1}{2}}$

i. $8x^3 + 7x^2 + 3x^3 - 9x^2 + 2$

j. $\frac{3}{4}n^{-3} + \frac{1}{2}n^3 - \frac{1}{2}n^{-3} + \frac{3}{8}n^3$

Solutions

a. $5x + 2x$

$= (5 + 2)x$

$= 7x$

b. $-6x^2 - 5x^2$

$= (-6 - 5)x^2$

$= -11x^2$

c. $6.2a^3 + 3.7a^3$

$= (6.2 + 3.7)a^3$

$= 9.9a^3$

d. $7a^{\frac{1}{2}} + 3a^{\frac{1}{2}} - 9a^{\frac{1}{2}}$

$= (7 + 3 - 9)\,a^{\frac{1}{2}}$

$= a^{\frac{1}{2}}$

e. $-5x^4 - 3x^4 - 2.5x^4$

$= (-5 - 3 - 2.5)x^4$

$= -10.5x^4$

f. $5x + 2y^{-5} + 3x + 7y^{-5}$

$= (5x + 3x) + (2y^{-5} + 7y^{-5})$

$= (5 + 3)x + (2 + 7)y^{-5}$

$= 8x + 9y^{-5}$

g. $4x^{-2} + 2x + 6x^{-2} + 7x$

$= (4x^{-2} + 6x^{-2}) + (2x + 7x)$

$= (4 + 6)x^{-2} + (2 + 7)x$

$= 10x^{-2} + 9x$

h. $7n^2 - 3n^{-\frac{1}{2}} + 2.6n^2 - 5.7n^{-\frac{1}{2}}$

$= \left(7n^2 + 2.6n^2\right) + \left(-3n^{-\frac{1}{2}} - 5.7n^{-\frac{1}{2}}\right)$

$= (7 + 2.6)\,n^2 + (-3 - 5.7)\,n^{-\frac{1}{2}}$

$= 9.6n^2 - 8.7n^{-\frac{1}{2}}$

i. $8x^3 + 7x^2 + 3x^3 - 9x^2 + 2$

$= (8 + 3)x^3 + (7 - 9)x^2 + 2$

$= 11x^3 - 2x^2 + 2$

j. $\frac{3}{4}n^{-3} + \frac{1}{2}n^3 - \frac{1}{2}n^{-3} + \frac{3}{8}n^3$

$= \left(\frac{1}{2}n^3 + \frac{3}{8}n^3\right) + \left(\frac{3}{4}n^{-3} - \frac{1}{2}n^{-3}\right)$

$= \left(\frac{1}{2} + \frac{3}{8}\right)n^3 + \left(\frac{3}{4} - \frac{1}{2}\right)n^{-3}$

$= \left(\frac{4+3}{8}\right)n^3 + \left(\frac{3-2}{4}\right)n^{-3}$

$= \frac{7}{8}n^3 + \frac{1}{4}n^{-3}$

2.2.2 Expansion and the Distributive Law

The Distributive Law can be written as:

$a(b + c) = ab + ac$

In algebra, we often need to ‘expand’ an expression, that is, to change it from a product to a sum. We apply the Distributive Law here in the same way it was applied earlier in the arithmetic.

Reminder: $5(3 + 1)$

$= 5 \times 3 + 5 \times 1$

$= 15 + 5$

$= 20$

The following examples illustrate the use of the Distributive Law in algebra.

Examples

Example

Expand $5(x + 1)$

$5(x + 1)$ — 5 is distributed to both terms inside the brackets

$= 5 \times x + 5 \times 1$

$= 5x + 5$

Example

Expand $5(x^3 + 3x^2 + 2)$

$5(x^3 + 3x^2 + 2)$ — 5 is distributed to all terms inside the brackets

$= 5 \times x^3 + 5 \times 3x^2 + 5 \times 2$

$= 5x^3 + 15x^2 + 10$

Example

Expand $2x(x^2 + 5x + 6)$

$2x(x^2 + 5x + 6)$ — $2x$ is distributed to all terms inside the brackets

$= 2x \times x^2 + 2x \times 5x + 2x \times 6$

$= 2x^3 + 10x^2 + 12x$

Exercise 2.3

a. $3(x + 1)$

b. $4(x + 2)$

c. $5(x - 3)$

d. $9(4 - 3n)$

e. $6(3 + 4a)$

f. $2x(3x - 5)$

g. $7x(6x^3 - 5x)$

h. $7x^2(2x^3 - 5a^2)$

i. $-2x^3(3y^3 - 2x^4)$

j. $-3x^2(4x^2 + 5x - 2)$

Solutions

a. $3(x + 1)$

$= 3 \times x + 3 \times 1$

$= 3x + 3$

b. $4(x + 2)$

$= 4 \times x + 4 \times 2$

$= 4x + 8$

c. $5(x-3)$

$= 5 \times x - 5 \times 3$

$= 5x - 15$

d. $9(4-3n)$

$= 9 \times 4 - 9 \times 3n$

$= 36 - 27n$

e. $6(3+4a)$

$= 6 \times 3 + 6 \times 4a$

$= 18 + 24a$

f. $2x(3x-5)$

$= 2x \times 3x - 2x \times 5$

$= 6x^2 - 10x$

g. $7x(6x^3 - 5x)$

$= 7x \times 6x^3 - 7x \times 5x$

$= 42x^4 - 35x^2$

h. $7x^2(2x^3 - 5a^2)$

$= 7x^2 \times 2x^3 - 7x^2 \times 5a^2$

(Note – alphabetical order in second term)

$= 14x^5 - 35a^2x^2$

i. $-2x^3(3y^3 - 2x^4)$

$= -2x^3 \times 3y^3 + 2x^3 \times 2x^4$ — distribute the $-2x^3$ to both terms

$= -6x^3y^3 + 4x^7$ — note that $-2x^3$ by $-2x^4$ gives a positive result

$= 4x^7 - 6x^3y^3$ — rearrange so that highest power is first

j. $-3x^2(4x^2 + 5x - 2)$

$= -3x^2 \times 4x^2 - 3x^2 \times 5x + 3x \times {}^2 2$ — distribute the $-3x^2$ to each term

$= -12x^4 - 15x^3 + 6x^2$ — note that $-3x^2$ by -2 gives a positive result

Another type of expression that you may need to expand is

$(x+3)\,(x+2)$

We do this by applying the same idea as before, that is everything inside the brackets is multiplied by what is outside. For example:

$(x+3)\,(x+2)$

$= x(x+2) + 3(x+2)$ — multiply the second bracket by everything in the first

$= x \times x + x \times 2 + 3 \times x + 3 \times 2$ — expand

$= x^2 + 2x + 3x + 6$

$= x^2 + (2+3)x + 6$ — group like terms

$= x^2 + 5x + 6$

Here are another two examples:

Examples

Example

Expand $(3x^2 + 2x)(4x^3 + 3x)$

$(3x^2 + 2x)(4x^3 + 3x)$	
$= 3x^2(4x^3 + 3x) + 2x(4x^3 + 3x)$	multiply the second bracket by everything in the first
$= 3x^2 \times 4x^3 + 3x^2 \times 3x + 2x \times 4x^3 + 2x \times 3x$	expand
$= 12x^5 + 9x^3 + 8x^4 + 6x^2$	

Example

Expand $(2x - 3)(4x^3 - 7x^2)$

$(2x - 3)(4x^3 - 7x^2)$	
$= (2x + -3)(4x^3 - 7x^2)$	rewrite –3 as '+ –3'
$= 2x(4x^3 - 7x^2) + -3(4x^3 - 7x^2)$	multiply the second bracket by everything in the first
$= 2 \times 4x^3 - 2x \times 7x^2 - 3 \times 4x^3 + 3 \times 7x^2$	expand
$= 8x^4 - 14x^3 - 12x^3 + 21x^2$	group like terms
$= 8x^4 + (-14 - 12)x^3 + 21x^2$	
$= 8x^4 - 26x^3 + 21x^2$	

Exercise 2.4

a. $(x + 1)(x + 2)$

b. $(x + 3)(x + 4)$

c. $(2x + 1)(3x + 4)$

d. $(5x^2 + 3)(4x^3 - 5)$

e. $(5 - 2x^2)(6x^3 - 7)$

Solutions

a. $(x + 1)(x + 2)$

$= x(x + 2) + 1(x + 2)$

$= x \times x + x \times 2 + x + 2$

$= x^2 + (2 + 1)x + 2$

$= x^2 + 3x + 2$

b. $(x + 3)\,(x + 4)$

$= x(x + 4) + 3(x + 4)$

$= x \times x + x \times 4 + 3 \times x + 3 \times 4$

$= x^2 + (4 + 3)x + 12$

$= x^2 + 7x + 12$

c. $(2x + 1)\,(\,3x + 4)$

$= 2x(3x + 4) + 1(3x + 4)$

$= 2x \times 3x + 2x \times 4 + 3x + 4$

$= 6x^2 + 8x + 3x + 4$

$= 6x^2 + (8 + 3)x + 4$

$= 6x^2 + 11x + 4$

d. $(5x^2 + 3)\,(4x^3 - 5)$

$= 5x^2(4x^3 - 5) + 3(4x^3 - 5)$

$= 5x^2 \times 4x^3 - 5x^2 \times 5 + 3 \times 4x^3 - 3 \times 5$

$= 20x^5 - 25x^2 + 12x^3 - 15$ rearrange into descending order of powers

$= 20x^5 + 12x^3 - 25x^2 - 15$

e. $(5 - 2x^2)\,(6x^3 - 7)$

$= (5 + -2x^2)\,(6x^3 - 7)$

$= 5(6x^3 - 7) + -2x^2(6x^3 - 7)$

$= 5 \times 6x^3 - 5 \times 7 - 2x^2 \times 6x^3 + 2x^2 \times 7$

$= 30x^3 - 35 - 12x^5 + 14x^2$

$= -12x^5 + 30x^3 + 14x^2 - 35$ rearrange into descending order of powers

Three factors may be expanded by expanding the last two of them and multiplying the result by the first. This may be extended to more than three factors if necessary.

Example

$$\begin{aligned}(2 + x)(3 - 2x)(1 + 2x) &= (2 + x)(3(1 + 2x) - 2x(1 + 2x))\\ &= (2 + x)(3 + 6x - 2x - 4x^2)\\ &= (2 + x)(3 + 4x - 4x^2)\\ &= 2(3 + 4x - 4x^2) + x(3 + 4x - 4x^2)\\ &= 6 + 8x - 8x^2 + 3x + 4x^2 - 4x^3\\ &= 6 + 11x - 4x^2 - 4x^3\end{aligned}$$

Exercise 2.5

Expand the following

a. $(x + 2)(x + 1)$

b. $(x - 2)(x + 1)$

c. $(x - 2)(x - 1)$

d. $(2x + 3)(x + 4)$

e. $(3x - 1)(2x + 4)$

f. $(2x - 5)(3x - 2)$

g. $(x - 1)(x + 1)(x + 2)$

h. $(2x - 1)(3x + 2)(4x + 1)$

i. $(x^2 - 3)(x + 2)$

j. $(x + y)(x - y)$

k. $(x + y)(x + y)$

l. $(x^2 + y^2)(x - y)(2x + y)$

Solutions

a. $x^2 + 3x + 2$

b. $x^2 - x - 2$

c. $x^2 - 3x + 2$

d. $2x^2 + 11x + 12$

e. $6x^2 + 10x - 4$

f. $6x^2 - 19x + 10$

g. $x^3 + 2x^2 - x - 2$

h. $24x^3 + 10x^2 - 7x - 2$

i. $x^3 + 2x^2 - 3x - 6$

j. $x^2 - y^2$

k. $x^2 + 2xy + y^2$

l. $2x^4 - x^3y + x^2y^2 - xy^3 - y^4$

2.2.3 Factorisation

The reverse process to expanding is called **factorisation**. In this case we are given an expression which is a sum of terms and we convert it to a product of two or more terms.

Expressions may be factorised by removing factors common to two or more terms and then by grouping and if necessary regrouping.

Examples

Example

$3x + 2x^2$	$= 3(x) + 2x(x)$	where x is the common factor
	$= x(3 + 2x)$	
$px^2 + qx^3$	$= p(x^2) + qx(x^2)$	
	$= x^2\,(p + qx)$	where x^2 is the common factor

$al + lb + am + mb$	$= l(a + b) + am + mb$	where l is the common factor
	$= l(a + b) + m(a + b)$	where m is the common factor of second two terms
	$= (a + b)\,(l + m)$	where $(a + b)$ is the common factor

$(a + b)^2 - a - b$	$= (a + b)^2 - (a + b)$	where -1 is the common factor
	$= (a + b)(a + b - 1)$	where $(a + b)$ is the common factor

If the expression is one of the form $ax^2 + bx + c$ then the above strategies are extended.

For example, when we expanded

$(x + 2)\,(x + 3)$

$= x(x + 3) + 2(x + 3)$

$= x^2 + 3x + 2x + 6$

$= x^2 + 5x + 6$

This means that $x^2 + 5x + 6$ is factorised to give $(x + 2)\,(x + 3)$, or $x^2 + 5x + 6$ has two factors, $(x + 2)$ and $(x + 3)$.

If we reverse through the process of expanding we do the following.

$x^2 + 5x + 6$	
$= x^2 + 3x + 2x + 6$	partition $5x$ into the sum of $3x + 2x$
$= x\,(x + 3) + 2\,(x + 3)$	take out common factors of x from the first two terms and 2 from the last two terms
$= (x + 3)\,(x + 2)$	take out a common factor of $(x + 3)$

To summarise expressions of the type $x^2 + bx + c$ can be factorised by using the result

$$(x + e)(x + f) = x^2 + (e + f)x + ef$$

Thus b is the sum of two numbers and c is the product.

Thus to factorise $x^2 + 7x + 6$, we need two numbers whose

sum = 7

product = 6

We select the appropriate numbers by guessing and checking. Solution is 6 and 1, giving factors $(x + 6)$ and $(x + 1)$.

For $x^2 - 7x + 6$, we need two numbers: sum = –7, product = 6

Solution is –6 and –1, giving factors $(x - 6)$ and $(x - 1)$.

For $x^2 + 2x - 3$ we need two numbers: sum = 2, product = –3

Solution is 3, –1, giving factors $(x + 3)$ and $(x - 1)$

Expressions of the type $ax^2 + bx + c$ can be factorised in the form $(dx + e)(fx + g)$. The constant a is the product df and c is the product eg. There are usually several possible numbers for d, f, e and g. Trial and error of the various combinations to obtain the one giving b is the only method.

For instance consider $6x^2 + 5x - 6$. The term $6x^2$ may be given by $3x$ and $2x$ or by $6x$ and x. The last term – 6, may be given by 6 and – 1, – 6 and 1, 3 and – 2 or – 3 and 2. The correct combination is $(3x - 2)(2x + 3)$.

Alternatively, we can make the coefficient of x^2 a one. This is achieved by factoring out 6 and then using the method described previously.

$$\begin{aligned}
&6x^2 + 5x - 6 \\
&= 6\left(x^2 + \frac{5}{6}x - 1\right) \\
&= 6\left[\left(x - \frac{2}{3}\right)\left(x + \frac{3}{2}\right)\right] \\
&= 2 \times 3\left[\left(x - \frac{2}{3}\right)\left(x + \frac{3}{2}\right)\right] \\
&= (3x - 2)(2x + 3)
\end{aligned}$$

Examples

Factorise:

a. $2ab - 10a + 3b - 15$

b. $y^2 - 5y - 36$

c. $6x^2 + 17x - 3$

Solutions

a. $2ab - 10a + 3b - 15 = 2a(b - 5) + 3(b - 5) = (b - 5)(2a + 3)$

b. We need two numbers whose product is -36 (so they will be of different sign) and whose sum is -5 (so the negative number will have the larger magnitude). The numbers are -9 and 4. Thus

$$y^2 - 5y - 36 = (y - 9)(y + 4)$$

c. We need two numbers whose product is 6, i.e. 6 and 1 or 3 and 2. We need two other numbers whose product is -3, i.e. 3 and -1 or -3 and 1. The correct combination of these to give $6x^2 + 17x - 3$ is $(6x - 1)(x + 3)$.

There are a number of standard factors that occur so often that they should be memorised. These are:

$$x^2 + 2xy + y^2 = (x + y)^2$$

$$x^2 - 2xy + y^2 = (x - y)^2$$

$$x^2 - y^2 = (x - y)(x + y)$$

Note: $x^2 + y^2$ cannot be factorised and $\sqrt{x^2 + y^2}$ is not $x + y$.

Exercise 2.6

Factorise:

a. $x^2 - x - 6$ b. $x^2 + 5x + 6$ c. $x^2 - 5x + 6$

d. $x^2 + x - 6$ e. $2x^2 - 3x - 14$ f. $6x^2 - 8x - 8$

g. $a^2b + ab^2 - abc$ h. $x^2 - 13x - 48$ i. $4x^2 - 8x + 3$

j. $8x^2 + 22x - 6$ k. $4 - 9x^2$ l. $15x^2 - 11x + 2$

Solutions

a. $(x - 3)(x + 2)$ b. $(x + 3)(x + 2)$ c. $(x - 3)(x - 2)$

d. $(x + 3)(x - 2)$ e. $(2x - 7)(x + 2)$ f. $2(x - 2)(3x + 2)$

g. $ab(a + b - c)$ h. $(x - 16)(x + 3)$ i. $(2x - 3)(2x - 1)$

j. $2(4x - 1)(x + 3)$ k. $(2 - 3x)(2 + 3x)$ l. $(5x - 2)(3x - 1)$

2.3 Algebraic fractions

In section 1.3.1 we looked at ways to calculate with numerical fractions. In this section we will use the same methods to perform operations with algebraic fractions.

2.3.1 Cancellation in fractions

In arithmetic $\frac{6}{9}$ and $\frac{2}{3}$ are equivalent fractions because $\frac{6}{9}$ can be written as $\frac{2\times 3}{3\times 3}$ and 3 which occurs in the numerator and denominator can be cancelled. This can be summarised by the expression.

$$\frac{am}{bm} = \frac{a}{b}$$

The same rules apply in algebra.

Consider

a. $\frac{2p}{rp} = \frac{2}{r}$ p is cancelled because it occurs top and bottom

b. $\frac{x(x+1)}{y(x+1)} = \frac{x}{y}$ $(x + 1)$ is cancelled

c. $\frac{(x+y)(a+b)}{z(x+y)} = \frac{(a+b)}{z}$ $(x + y)$ is cancelled

Important: Notice that in the fraction $\frac{am}{bm}$, the numerator **and** the denominator are both products.

There is no cancellation rule for sums

i.e. in $\frac{a+m}{b+m}$, m cannot be cancelled.

Exercise 2.7

a. $\frac{x}{2}, \frac{6x}{12}$

b. $\frac{x}{2}, \frac{2x}{4x}$

c. $\frac{x+1}{x-2}, \frac{2(x+1)}{2(x-2)}$

d. $\frac{x+1}{y+1}, \frac{x}{y}$

e. $\frac{(x-5)(x^2+2y)}{(x^2+2y)(x+5)}, \frac{x-5}{x+5}$

f. $\frac{2(x-3y)}{x-3y}, 2$

Solutions

a. $\frac{x}{2} = \frac{6x}{6\times 2} = \frac{6x}{12}$ equivalent to $\frac{6x}{12}$

b. $\frac{x}{2} = \frac{2x}{2\times 2} = \frac{2x}{4}$ is not equivalent to $\frac{2x}{4x}$

c. $\frac{x+1}{x-2} = \frac{2(x+1)}{2(x-2)}$ equivalent

d. $\frac{x+1}{y+1}$ cannot cancel 1 because of sum not product, not equivalent to $\frac{x}{y}$

e. $\frac{(x-5)(x^2+2y)}{(x^2+2y)(x+5)} = \frac{x-5}{x+5}$ cancel (x^2+2y), equivalent

f. $\frac{2(x-3y)}{x-3y} = \frac{2(x-3y)}{1\times(x-3y)} = \frac{2}{1} = 2$, cancel $(x-3y)$ equivalent

2.3.2 Addition and subtraction of fractions

Using rules developed for numerical fractions in section 1.3.1 we can develop the following patterns for addition and subtraction of fractions.

Examples

Examples

a. $\dfrac{x}{y}+\dfrac{m}{y}=\dfrac{x+m}{y}$ because denominators are the same

b. $\dfrac{x}{z}+\dfrac{y}{k}=\dfrac{xk+yz}{zk}$ find the lowest common denominator because denominators are different

c. $\dfrac{2}{x}+\dfrac{z}{xy}=\dfrac{2y}{xy}+\dfrac{z}{xy}$ find the lowest common denominator

$$=\frac{2y+z}{xy}$$

Example

Simplify $\dfrac{x}{x-3}+\dfrac{5}{x+2}$

$$\frac{x}{x-3}+\frac{5}{x+2}=\frac{x(x+2)+5(x-3)}{(x-3)(x+2)}=\frac{x^2+2x+5x-15}{x^2+2x-3x-6}=\frac{x^2+7x-15}{x^2-x-6}$$

Example

Simplify $\dfrac{x+2}{x-2}-\dfrac{x-2}{x+2}$

$$\frac{x+2}{x-2}-\frac{x-2}{x+2}=\frac{(x+2)^2-(x-2)^2}{(x-2)(x+2)}=\frac{(x^2+4x+4)-(x^2-4x+4)}{x(x+2)-2(x+2)}$$

$$=\frac{x^2+4x+4-x^2+4x-4}{x^2+2x-2x-4}=\frac{8x}{x^2-4}$$

Exercise 2.8

Simplify the following expressions:

a. $\frac{x+1}{x-2}+\frac{x-1}{x+2}$ b. $\frac{x-3}{x-1}-\frac{2x-1}{x+2}$ c. $\frac{3x-1}{x+2}-\frac{2x+1}{x-3}$

d. $\frac{x}{2x+1}+\frac{x-1}{3x-1}$ e. $\frac{4}{x-1}+\frac{x-2}{x+3}$ f. $\frac{2x-3}{x+2}-\frac{x-1}{x-3}$

Solutions

a. $\frac{2x^2+4}{x^2-4}$ b. $\frac{-x^2+2x-7}{x^2+x-2}$ c. $\frac{x^2-15x+1}{x^2-x-6}$

d. $\frac{5x^2-2x-1}{6x^2+x-1}$ e. $\frac{x^2+x+14}{x^2+2x-3}$ f. $\frac{3x^2-8x+7}{x^2-x-6}$

2.4 Solving equations and inequations

So far we have looked only at algebraic expressions but in section 2.1 we generated expressions that were actually equations. An equation is a representation of two expressions which are equal. Equations can come in a range of forms depending on the expressions involved, the simplest is the linear equation.

2.4.1 Linear equations

Linear equations are equations which contain variables of a power of 1 only, i.e. variables would be x, y or z, not xy, x^2 or z^3.

$x + y = 5$ is a linear equation

$3x - 4y = -1$ is a linear equation

$x^2 + 2 = 5$ is not a linear equation

$xy + z = 7$ is not a linear equation

(See section 3.2.1 for more information on linear functions)

One of the main aims of constructing equations is to find the value of a variable that will solve the equation.

Consider the equation $x + 5 = 27$. With some thought you can probably say that in this case, x must have been 22. Other equations may be more difficult to solve.

We are now going to develop a set of procedures to use to solve any type of linear equation.

The key point to remember when solving equations is to **isolate the variable**. To do this:

1. Remove all constants.
2. Make the coefficient in front of the variable one.

Use the opposite operation to achieve cancelling. A reminder:

Operation	Opposite
+	–
–	+
×	÷
÷	×

It is essential to maintain the balance of the equation throughout. Therefore, **whatever you do to one side you <u>must</u> do to the other**.

Let's use these ideas to solve $x + 5 = 27$.

$$x + 5 = 27$$

$$x + 5 - 5 = 27 - 5$$

Cancel the 5 on the LHS (left hand side) by subtracting 5. Whatever you do to the LHS do to the RHS (right hand side)

$$x = 22$$

When there is more than one operation involved, e.g. $2x + 3 = 5$, they are cancelled out in the **reverse** of Order of Operations, that is:

1. + and –
2. × and ÷
3. brackets

Below are some samples to illustrate the techniques involved in solving different types of linear equations. They provide models for setting out all of the basic types you will encounter in this course. It is **not** intended that they be learnt as rules – use them as a reference until you no longer need them. (Note that we check our answer at the end by substituting it for the variable in the original equation.)

Number	Type	Example	Steps involved
1	$x + a = b$	$x + 9 = 12$ $x + 9 - 9 = 12 - 9$ $x = 3$	By subtracting 9 from the LHS we cancel the given 9. To keep the balance do the same to the RHS. **Check**: LHS = 3 + 9 = 12 = RHS
2	$x - a = b$	$x - 2 = -7$ $x - 2 + 2 = -7 + 2$ $x = -5$	Add 2 to the LHS to cancel the given 2. Add 2 to the RHS to keep the balance. **Check**: LHS = –5 – 2 = – 7 = RHS
3	$ax = b$	$3x = 29$ $\frac{3x}{3} = \frac{29}{3}$ $x = 9\frac{2}{3}$ $x \approx 9.67$	Cancel the 3 on the LHS by dividing by 3. Do the same to the RHS. Coefficient of x is now 1. Express answer as either a mixed number or a decimal. **Check**: LHS = $3 \times 9\frac{2}{3} = 3 \times \frac{29}{3} = 29 =$ RHS
4	$\frac{x}{a} = b$	$\frac{x}{5} = -4$ $\frac{x}{5} \times 5 = -4 \times 5$ $x = -20$	Multiply the LHS by 5 to cancel the given 5. Maintain the balance by multiplying the RHS by 5. **Check**: LHS = $\frac{-20}{5} = -4 =$ RHS
5	$a + x = b$	$3 + x = 8.5$ $3 + x - 3 = 8.5 - 3$ $x = 5.5$	Subtract 3 from both sides. **Check**: LHS = 3 + 5.5 = 8.5 = RHS
6	$a - x = b$	$7 - x = 2$ $7 - x - 7 = 2 - 7$ $-x = -5$ $-x \times -1 = -5 \times -1$ $x = 5$	Coefficient of x is –1. Multiply LHS by –1 to give a coefficient of 1. Do the same for the RHS. **Check**: LHS = 7 – 5 = 2 = RHS
7	$ax + b = c$	$3x + 8 = 14$ $3x + 8 - 8 = 14 - 8$ $3x = 6$ $\frac{3x}{3} = \frac{6}{3}$ $x = 2$	Cancel out addition first. Then cancel multiplication. **Check**: LHS = 3 × 2 + 8 = 6 + 8 = 14 = RHS
8	$b + ax = c$	$3 + 7x = 17$ $3 + 7x - 3 = 17 - 3$ $7x = 14$ $\frac{7x}{7} = \frac{14}{7}$ $x = 2$	Cancel addition. Cancel multiplication. **Check**: LHS = 3 + 7 × 2 = 3 + 14 = 17 = RHS

Number	Type	Example	Steps involved
9	$b - ax = c$	$18-5x=17$ $18-5x-18=17-18$ $-5x=-1$ $\frac{-5x}{-5}=\frac{-1}{-5}$ $x=\frac{1}{5}$	Cancel 18 on LHS. Cancel –5 on LHS. **Check**: LHS $= 18-5\times\frac{1}{5}=18-1=17=$ RHS
10	$ax + b + cx = d$	$2x+8+3x=3$ $2x+3x+8=3$ $(2+3)x+8=3$ $5x+8=3$ $5x+8-8=3-8$ $5x=-5$ $\frac{5x}{5}=\frac{-5}{5}$ $x=-1$	Group like terms. Cancel 8 on LHS Cancel 5 on LHS **Check**: LHS = 2 × –1 + 8 + 3 × –1 = –2 + 8 – 3 = 3 = RHS
11	$a(x + b) = c$	$3(x+1)=12$ $\frac{3(x+1)}{3}=\frac{12}{3}$ $x+1=4$ $x+1-1=4-1$ $x=3$	Leave brackets. Cancel 3 on LHS Now cancel 1 on LHS **Check**: LHS = 3(3 + 1) = 3 × 4 = 12 = RHS
12	$ax + b = cx + d$	$3x+2=5x+4$ $3x+2-5x=5x+4-5x$ $3x-5x+2=4$ $(3-5)x+2=4$ $-2x+2=4$ $-2x+2-2=4-2$ $-2x=2$ $\frac{-2x}{x}=\frac{2}{-2}$ $x=-1$	Group like terms on LHS Cancel 2 on LHS Cancel –2 on LHS **Check**: LHS = 3(–1) + 2 = –1 RHS = 5(–1) + 4 = –1 LHS = RHS

Exercise 2.9

Solve: Verify your answer each time by substituting it into the original equation.

a. $x + 3 = 9$

b. $x - 2 = 5$

c. $5x = 15$

d. $\frac{x}{2} = 6$

e. $-3 + x = 7$

f. $2.4 - x = 3.6$

g. $2x + 1 = -8$

h. $\frac{x}{5} - 2 = 7$

i. $5 + 2x = 1$

j. $10 - 3x = 1$

k. $\frac{3}{4}(x + 3) = \frac{3}{8}$

l. $2.5x + 3.5 = 7$

m. $4\left(3x - \frac{4}{9}\right) = 3$

n. $2x + 3 + 5x = 17$

o. $\frac{3x}{4} - \frac{2}{3} + \frac{x}{2} = \frac{5}{6}$

p. $\frac{4x}{3} = 7 + \frac{x}{6}$

q. $\frac{4x}{5} + \frac{2}{3} = 1$

r. $5x + 3 - 2x = 21$

s. $2x + 1 = 3x - 5$

t. $0.5x - 4.1 = 3.9 - 0.7x$

Solutions

a.
$$x + 3 = 9$$
$$x + 3 - 3 = 9 - 3$$
$$x = 6$$

Isolate x by taking 3 from LHS. Also take 3 from the RHS.

Check: LHS = 6 + 3 = 9 = RHS

b.
$$x - 2 = 5$$
$$x - 2 + 2 = 5 + 2$$
$$x = 7$$

Isolate x by adding 2 to LHS. Do the same to RHS.

Check: LHS = 7 – 2 = 5 = RHS

c.
$$5x = 15$$
$$\frac{5x}{5} = \frac{15}{5}$$
$$x = 3$$

Isolate x by dividing LHS by 5.

Do the same to RHS.

Check: LHS = 5 × 3 = 15 = RHS

d.
$$\frac{x}{2} = 6$$
$$\frac{x}{2} \times 2 = 6 \times 2$$
$$x = 12$$

Isolate x by multiplying LHS by 2.

Do the same to RHS.

Check: LHS = $\frac{12}{2}$ = 6 = RHS

e.

$$-3+x=7$$
$$-3+x+3=7+3$$
$$x=10$$

Isolate *x* by adding 3 to LHS. Do the same to RHS

Check: LHS = –3 + 10 = 7 = RHS

f.

$$2.4-x=3.6$$
$$2.4-x-2.4=3.6-2.4$$
$$-x=1.2$$
$$x=-1.2$$

Isolate *x* by taking 2.4 from LHS. Do the same to RHS. Multiply both sides by –1 to make coefficient of the *x* a one.

Check: LHS= 2.4 – (–1.2) = 2.4 + 1.2 = 3.6 = RHS

g.

$$2x+1=-8$$
$$2x+1-1=-8-1$$
$$2x=-9$$
$$\frac{2x}{2}=\frac{-9}{2}$$
$$x=-4\frac{1}{2}$$

Isolate *x* by taking 1 from LHS. Do the same to RHS.

Divide LHS by 2 to make coefficient of *x* a one. Do the same to RHS.

Check: LHS =

$$2\times\left(-4\frac{1}{2}\right)+1=2\times\frac{-9}{2}+1=-9+1=-8=\text{ RHS}$$

h.

$$\frac{x}{5}-2=7$$
$$\frac{x}{5}-2+2=7+2$$
$$\frac{x}{5}=9$$
$$\frac{x}{5}\times 5=9\times 5$$
$$x=45$$

Isolate *x* by adding 2 to LHS. Do the same to RHS

Multiply LHS by 5. Do the same to RHS

Check: LHS = $\frac{45}{5}-2=9-2=7=$ RHS

i.

$$5+2x=1$$
$$5+2x-5=1-5$$
$$2x=-4$$
$$\frac{2x}{2}=\frac{-4}{2}$$
$$x=-2$$

Take 5 from LHS to isolate the *x*.

Do the same to RHS.

Divide LHS by 2 to make coefficient of *x* a one. Do the same to RHS.

Check: LHS = 5 + 2 × –2 = 5 – 4 = 1 = RHS

j.

$$10-3x=1$$
$$10-3x-10=1-10$$
$$-3x=-9$$
$$\frac{-3x}{-3}=\frac{-9}{-3}$$
$$x=3$$

Take 10 from LHS to isolate the *x*.

Do the same to RHS.

Divide LHS by –3 to make coefficient of *x* a one. Do the same to RHS.

Check: LHS = 10 – 3 × 3 = 10 – 9 = 1 = RHS k.

$$\frac{3}{4}(x+3) = \frac{3}{8}$$

$$\frac{3}{4} \times \frac{4}{3}(x+3) = \frac{3}{8} \times \frac{4}{3}$$

Remove factor outside bracket on LHS by multiplying by its reciprocal. Do the same to RHS. Simplify RHS by cancelling common factors.

$$x+3 = \frac{1}{2}$$

Take 3 from LHS. Do same to RHS.

$$x+3-3 = \frac{1}{2}-3$$

Find a common denominator on RHS.

$$x = \frac{1-6}{2}$$

Express answer as a mixed number.

$$x = \frac{-5}{2}$$

Check: LHS = $\frac{3}{4}\left(-2\frac{1}{2}+3\right) = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8} =$ RHS

$$x = -2\frac{1}{2}$$

l.

$$2.5x + 3.5 = 7$$

$$2.5x + 3.5 - 3.5 = 7 - 3.5$$

subtract 3.5 from LHS. Do same to RHS.

$$2.5x = 3.5$$

divide LHS by 2.5. Do same to RHS.

$$\frac{2.5x}{2.5} = \frac{3.5}{2.5}$$

Check: LHS = 2.5 × 1.4 + 3.5 = 3.5 + 3.5 = 7 = RHS

$$x = 1.4$$

m.

$$4\left(3x - \frac{4}{9}\right) = 3$$

$$\frac{4}{4}\left(3x - \frac{4}{9}\right) = \frac{3}{4}$$

divide LHS by 4. Do same to RHS.

$$3x - \frac{4}{9} = \frac{3}{4}$$

$$3x - \frac{4}{9} + \frac{4}{9} = \frac{3}{4} + \frac{4}{9}$$

add $\frac{4}{9}$ to LHS. Do same to RHS.

$$3x = \frac{27+16}{36}$$

find a common denominator on RHS.

$$3x = \frac{43}{36}$$

$$\frac{3x}{3} = \frac{43}{36} \times \frac{1}{3}$$

divide LHS by 3. Do same to RHS.

$$x = \frac{43}{108}$$

Check:

$$\text{LHS} = 4\left(3 \times \frac{43}{108} - \frac{4}{9}\right) = 4\left(\frac{129}{108} - \frac{4}{9}\right) = 4\left(\frac{129-48}{108}\right)$$

$$= 4\left(\frac{81}{108}\right) = 4 \times \frac{3}{4} = 3 = \text{RHS}$$

n.
$$2x+3+5x=17$$
$$2x+5x+3=17$$
$$(2+5)x+3=17$$ group like terms
$$7x+3=17$$
$$7x+3-3=17-3$$
$$7x=14$$
$$\frac{7x}{7}=\frac{14}{7}$$
$$x=2$$

Check: LHS = 2 × 2 + 3 + 5 × 2 = 4 + 3 + 10 = 17 = RHS

o.
$$\frac{3x}{4}-\frac{2}{3}+\frac{x}{2}=\frac{5}{6}$$
$$\frac{3x}{4}+\frac{x}{2}-\frac{2}{3}=\frac{5}{6}$$ group like terms
$$\frac{3x+2x}{4}-\frac{2}{3}=\frac{5}{6}$$ find a common denominator on LHS
$$\frac{5x}{4}-\frac{2}{3}=\frac{5}{6}$$
$$\frac{5x}{4}-\frac{2}{3}+\frac{2}{3}=\frac{5}{6}+\frac{2}{3}$$
$$\frac{5x}{4}=\frac{5+4}{6}$$ find a common denominator on RHS
$$\frac{5x}{4}=\frac{9}{6}$$
$$\frac{5x}{4}=\frac{3}{2}$$
$$\frac{5x}{4}\times\frac{4}{5}=\frac{3}{2}\times\frac{4}{5}$$
$$x=\frac{6}{5}$$
$$x=1\frac{1}{5}$$

Check:
$$\text{LHS}=\frac{3}{4}\times\frac{6}{5}-\frac{2}{3}+\frac{1}{2}\times\frac{6}{5}=\frac{9}{10}-\frac{2}{3}+\frac{6}{10}$$
$$=\frac{27-20+18}{30}=\frac{25}{30}=\frac{5}{6}=\text{RHS}$$

p.

$$\frac{4x}{3} = 7 + \frac{x}{6}$$

$$\frac{4x}{3} - \frac{x}{6} = 7 + \frac{x}{6} - \frac{x}{6}$$ group like terms

$$\frac{8x - x}{6} = 7$$

$$\frac{7x}{6} = 7$$

$$\frac{7x}{6} \times \frac{6}{1} = 7 \times 6$$

$$7x = 42$$

$$\frac{7x}{7} = \frac{42}{7}$$

$$x = 6$$

Check: LHS $= \frac{4 \times 6}{3} = 8$

RHS $= 7 + \frac{6}{6} = 7 + 1 = 8 =$ LHS

q.

$$\frac{4x}{5} + \frac{2}{3} = 1$$

$$\frac{4x}{5} + \frac{2}{3} - \frac{2}{3} = 1 - \frac{2}{3}$$

$$\frac{4x}{5} = \frac{3 - 2}{3}$$

$$\frac{4x}{5} = \frac{1}{3}$$

$$\frac{4x}{5} \times \frac{5}{4} = \frac{1}{3} \times \frac{5}{4}$$

$$x = \frac{5}{12}$$

Check: LHS $= \frac{4}{5} \times \frac{5}{12} + \frac{2}{3} = \frac{1}{3} + \frac{2}{3} = 1 =$ RHS

r.

$$5x + 3 - 2x = 21$$

$$5x - 2x + 3 = 21$$

$$3x + 3 = 21$$

$$3x + 3 - 3 = 21 - 3$$

$$3x = 18$$

$$\frac{3x}{3} = \frac{18}{3}$$

$$x = 6$$

Check: LHS = $5 \times 6 + 3 - 2 \times 6 = 30 + 3 - 12 = 21 =$ RHS s.

$$\begin{aligned}
2x+1 &= 3x-5\\
2x+1-3x &= 3x-5-3x\\
2x-3x+1 &= -5\\
-x+1 &= -5\\
-x+1-1 &= -5-1\\
-x &= -6\\
x &= 6
\end{aligned}$$

Check: LHS = 2 × 6 + 1 = 12 + 1 = 13
RHS = 3 × 6 – 5 = 18 – 5 = 13 = LHS

t.

$$\begin{aligned}
0.5x-4.1 &= 3.9-0.7x\\
0.5x-4.1+0.7x &= 3.9-0.7x+0.7x\\
0.5x+0.7x-4.1 &= 3.9\\
1.2x-4.1 &= 3.9\\
1.2x-4.1+4.1 &= 3.9+4.1\\
1.2x &= 8\\
\frac{1.2x}{1.2} &= \frac{8}{1.2}\\
x &\approx 6.67\\
\text{or } x &= 6\frac{2}{3}
\end{aligned}$$

Check:
LHS = 0.5 × 6.67 – 4.1 = 3.335 – 4.1 = –0.765
RHS = 3.9 – 0.7 × 6.67 = 3.9 – 4.669 = –0.769

2.4.2 Rearranging formulae

A **formula** is a rule connecting different variables. For example, the area of a rectangle is found by multiplying its length by its width, that is,

$$A = lw$$

This is a formula, where A, l and w are variables.

The **subject of the formula** is the variable on its own, usually on the left hand side, e.g. A is the subject of the formula above.

Sometimes we know the value of the subject but not that of one of the other variables. In such cases, we need to rearrange the formula so that the unknown variable becomes the 'subject of the formula'. This is why this is an important algebraic skill to master.

There are a number of operations which can be used to rearrange formulae. These include:

- adding or subtracting terms from both sides
- multiplying or dividing throughout
- changing the sign throughout
- raising both sides to the same power (e.g. squaring or taking the square root)
- taking reciprocals of both sides.

Here is an example of a situation where rearranging of the formula is needed.

To calculate speed (s) we divide distance (d) by time (t).

This is shown by the formula: $s = \frac{d}{t}$

Now suppose you wish to estimate the length of time it will take you to travel 100 km at an average speed of 90 km per hour. (Note: km per hour is usually written as km h^{-1}). Time is not the subject of the formula. You need to transpose (rearrange) to make it so:

$$s = \frac{d}{t}$$

$$ts = \frac{dt}{t}$$

$$ts = d$$

$$\frac{ts}{s} = \frac{d}{s}$$

$$t = \frac{d}{s}$$

It is now in the form we need. Let's substitute to find the time it would take:

$$t = \frac{d}{s}$$

$$t = \frac{100 \text{ km}}{90 \text{ km h}^{-1}}$$ substitute $d = 100$ and $s = 90$

$$t = \frac{100 \text{ h}}{90}$$ recall what negative powers mean

$$= 1\frac{1}{9} \text{ h}$$ Make sure the units cancel to give the appropriate unit for the answer.

Examples

Example

Make n the subject of this formula

$$T = 5(n + 40)$$

$$\begin{aligned} T &= 5(n+40) \\ \frac{T}{5} &= \frac{5(n+40)}{5} && \text{note that the 5 is moved first} \\ \frac{T}{5} &= n+40 \\ \frac{T}{5} - 40 &= n + 40 - 40 \\ \frac{T}{5} - 40 &= n \\ n &= \frac{T}{5} - 40 && \text{put the subject on LHS} \end{aligned}$$

Example

Make n the subject of this formula

$$\begin{aligned} T &= (3n+1)^2 \\ \pm\sqrt{T} &= 3n+1 && \text{take the square root of both sides} \\ \pm\sqrt{T} - 1 &= 3n && \text{both the positive and negative root must be given} \\ \frac{\pm\sqrt{T}-1}{3} &= n \\ n &= \frac{\pm\sqrt{T}-1}{3} \end{aligned}$$

Example

Make n the subject of this formula

$$y = \sqrt{4n^2 + 3}$$

$$y^2 = 4n^2 + 3 \qquad \text{square both sides}$$

$$y^2 - 3 = 4n^2$$

$$\frac{y^2 - 3}{4} = n^2$$

$$\pm\sqrt{\frac{y^2 - 3}{4}} = n \qquad \text{take the square root of both sides}$$

$$n = \pm\sqrt{\frac{y^2 - 3}{4}}$$

$$n = \frac{\pm\sqrt{y^2 - 3}}{2} \qquad \text{simplify by finding } \sqrt{4}$$

Example

Make p the subject of this formula

$$k = \sqrt{\frac{f + p}{f - p}}$$

$$\therefore\ k^2 = \frac{f + p}{f - p} \qquad \text{square both sides}$$

$$\therefore\ k^2(f - p) = f + p \qquad \text{multiply by } (f - p)$$

$$\therefore\ k^2 f - k^2 p = f + p$$

$$\therefore\ -p + k^2 f - k^2 p = f \qquad \text{subtract } p \text{ from both sides}$$

$$\therefore\ -p - k^2 p = f - k^2 f \qquad \text{subtract } k^2 f \text{ from both sides}$$

$$\therefore\ p + k^2 p = -f + k^2 f \qquad \text{change sign throughout}$$

$$\therefore\ p\left(1 + k^2\right) = -f + k^2 f$$

$$\therefore\ p = \frac{-f + k^2 f}{1 + k^2} \qquad \text{divide by } (1 + k^2)$$

Example

If an object is placed at a distance u from a spherical mirror of radius r, its image appears at a distance from the mirror. It can be shown that: $\frac{1}{v}+\frac{1}{u}=\frac{2}{r}$.

Make r the subject of the formula.

$$\frac{1}{v}+\frac{1}{u}=\frac{2}{r}$$

$$\therefore\ \frac{u+v}{vu}=\frac{2}{r}$$

$$\therefore\ \frac{vu}{u+v}=\frac{r}{2}$$ reciprocal of both sides

$$\therefore\ r=\frac{2vu}{u+v}$$ multiply by 2

Example

The oscillating time, t, of a pendulum of length L is given by: $t=2\pi\sqrt{\frac{L}{g}}$, where g is the acceleration due to gravity. Find an expression for L.

$$t=2\pi\sqrt{\frac{L}{g}}$$

$$\frac{t}{2\pi}=\sqrt{\frac{L}{g}}$$ divided by 2π

$$\therefore\ \left(\frac{t}{2\pi}\right)^2=\frac{L}{g}$$ square both sides

$$\therefore\ \frac{t^2}{4\pi^2}=\frac{L}{g}$$

$$\therefore\ L=\frac{t^2 g}{4\pi^2}$$ multiply by g

Exercise 2.10

Make b the subject of the formula

a. $x = 5b$

b. $a = 6 + b$

c. $k = \dfrac{5}{b}$

d. $c = 3(5b + 1)$

e. $m = 6b^2 - 2$

f. $d = (3b + 2)^2$

g. $A = \dfrac{bh}{2}$

h. $A = \dfrac{(a + b)h}{2}$

i. $x = 3b - \dfrac{1}{9}$

j. $y = 3.1b + 2.4$

k. $K = \sqrt{b^2 + 4}$

l. $m = \dfrac{7 + 2b^2}{2}$

m. $m = \dfrac{\sqrt{b^2 + 2}}{5}$

n. $r = \dfrac{b^3}{5} + 3$

o. $y = \left(\dfrac{6b^2 + 2}{7}\right)^2$

Solutions

a.
$$x = 5b$$
$$\frac{x}{5} = \frac{5b}{5}$$
$$\frac{x}{5} = b$$
$$b = \frac{x}{5}$$

b.
$$a = 6 + b$$
$$a - 6 = 6 + b - 6$$
$$a - 6 = b$$
$$b = a - 6$$

c.

$$\begin{aligned} k &= \frac{5}{b} \\ kb &= \frac{5}{b} \times b \\ kb &= 5 \\ \frac{kb}{k} &= \frac{5}{k} \\ b &= \frac{5}{k} \end{aligned}$$

d.

$$\begin{aligned} c &= 3(5b+1) \\ \frac{c}{3} &= \frac{3(5b+1)}{3} \\ \frac{c}{3} &= 5b+1 \\ \frac{c}{3} - 1 &= 5b+1-1 \\ \frac{c}{3} - 1 &= 5b \\ \frac{\frac{c}{3}-1}{5} &= \frac{5b}{5} \\ \frac{\frac{c}{3}-1}{5} &= b \\ b &= \frac{\frac{c}{3}-1}{5} \end{aligned}$$

e.

$$\begin{aligned} m &= 6b^2 - 2 \\ m+2 &= 6b^2 - 2 + 2 \\ m+2 &= 6b^2 \\ \frac{m+2}{6} &= \frac{6b^2}{6} \\ \pm\sqrt{\frac{m+2}{6}} &= b \\ b &= \pm\sqrt{\frac{m+2}{6}} \end{aligned}$$

f.

$$\begin{aligned} d &= (3b+2)^2 \\ \pm\sqrt{d} &= 3b+2 \\ \pm\sqrt{d} - 2 &= 3b+2-2 \\ \pm\sqrt{d} - 2 &= 3b \\ \frac{\pm\sqrt{d}-2}{3} &= \frac{3b}{3} \\ \frac{\pm\sqrt{d}-2}{3} &= b \\ b &= \frac{\pm\sqrt{d}-2}{3} \end{aligned}$$

g.

$$\begin{aligned} A &= \frac{bh}{2} \\ \frac{2A}{h} &= \frac{bh}{2} \times \frac{2}{h} \\ \frac{2A}{h} &= b \\ b &= \frac{2A}{h} \end{aligned}$$

h.

$$\begin{aligned} A &= \frac{(a+b)h}{2} \\ \frac{2A}{h} &= \frac{(a+b)h}{2} \times \frac{2}{h} \\ \frac{2A}{h} &= a+b \\ \frac{2A}{h} - a &= b \\ b &= \frac{2A}{h} - a \end{aligned}$$

i.

$$\begin{aligned} x &= 3b - \frac{1}{9} \\ x + \frac{1}{9} &= 3b - \frac{1}{9} + \frac{1}{9} \\ x + \frac{1}{9} &= 3b \\ \frac{x + \frac{1}{9}}{3} &= \frac{3b}{3} \\ b &= \frac{x + \frac{1}{9}}{2} = \frac{x}{3} + \frac{1}{27} \end{aligned}$$

j.

$$\begin{aligned} y &= 3.1b + 2.4 \\ y - 2.4 &= 3.1b + 2.4 - 2.4 \\ y - 2.4 &= 3.1b \\ \frac{y - 2.4}{3.1} &= b \\ b &= \frac{y - 2.4}{3.1} \end{aligned}$$

k.

$$\begin{aligned} K &= \sqrt{b^2 + 4} \\ K^2 &= b^2 + 4 \\ K^2 - 4 &= b^2 + 4 - 4 \\ K^2 - 4 &= b^2 \\ \pm\sqrt{K^2 - 4} &= b \\ b &= \pm\sqrt{K^2 - 4} \end{aligned}$$

l.

$$\begin{aligned} m &= \frac{7 + 2b^2}{2} \\ 2m &= \frac{\left(7 + 2b^2\right)}{2} \times 2 \\ 2m &= 7 + 2b^2 \\ 2m - 7 &= 7 + 2b^2 - 7 \\ 2m - 7 &= 2b^2 \\ \frac{2m - 7}{2} &= \frac{2b^2}{2} \\ \frac{2m - 7}{2} &= b^2 \\ \pm\sqrt{\frac{2m - 7}{2}} &= b \\ b &= \pm\sqrt{\frac{2m - 7}{2}} \end{aligned}$$

m.

$$m=\frac{\sqrt{b^2+2}}{5}$$
$$5m=\frac{\sqrt{b^2+2}}{5}\times\frac{5}{1}$$
$$5m=\sqrt{b^2+2}$$
$$(5m)^2=b^2+2$$
$$25m^2=b^2+2$$
$$25m^2-2=b^2+2-2$$
$$25m^2-2=b^2$$
$$\pm\sqrt{25m^2-2}=b$$
$$b=\pm\sqrt{25m^2-2}$$

n.

$$r=\frac{b^3}{5}+3$$
$$r-3=\frac{b^3}{5}+3-3$$
$$r-3=\frac{b^3}{5}$$
$$5(r-3)=\frac{b^3}{5}\times\frac{5}{1}$$
$$5r-5\times 3=b^3$$
$$5r-15=b^3$$
$$(5r-15)^{\frac{1}{3}}=\left(b^3\right)^{\frac{1}{3}}$$
$$(5r-15)^{\frac{1}{3}}=b$$
$$b=(5r-15)^{\frac{1}{3}}$$
$$\text{or } b=\sqrt[3]{5r-15}$$

o.

$$y=\left(\frac{6b^2+2}{7}\right)^2$$
$$\sqrt{y}=\frac{6b^2+2}{7}$$
$$7\sqrt{y}=\frac{\left(6b^2+2\right)}{7}\times 7$$
$$7\sqrt{y}=6b^2+2$$
$$7\sqrt{y}-2=6b^2+2-2$$
$$7\sqrt{y}-2=6b^2$$
$$\frac{7\sqrt{y}-2}{6}=\frac{6b^2}{6}$$
$$\frac{7\sqrt{y}-2}{6}=b^2$$
$$\pm\sqrt{\frac{7y^{\frac{1}{2}}-2}{6}}=b$$
$$b=\pm\sqrt{\frac{7y^{\frac{1}{2}}-2}{6}}$$

Exercise 2.11

Transform the following equations to make the stated variable the subject.

a. $s = rt$; r

b. $c = 2\pi r$; r

c. $e = mc^2$; m

d. $e = mc^2$; c

e. $F = \dfrac{Gm_1m_2}{d}$; G

f. $\dfrac{1}{f} = \dfrac{1}{a} + \dfrac{1}{b}$; f

g. $P = \dfrac{1}{3}Nmv^2$; v

h. $V = \dfrac{1}{2rL}\sqrt{\dfrac{T}{\pi d}}$; T

i. $3x - 4y - 12 = 0$; y

j. $y = \dfrac{2x - 3}{3x - 5}$; x

Solutions

a. $r = \dfrac{s}{t}$

b. $r = \dfrac{c}{2\pi}$

c. $m = \dfrac{e}{c^2}$

d. $c = \mp\sqrt{\dfrac{e}{m}}$

e. $G = \dfrac{Fd}{m_1m_2}$

f. $f = \dfrac{ab}{a + b}$

g. $v = \pm\sqrt{\dfrac{3P}{Nm}}$

h. $T = 4\pi r^2 L^2 V^2 d$

i. $y = \dfrac{3x - 12}{4}$

j. $x = \dfrac{5y - 3}{3y - 2}$

2.4.3 Quadratic equations

A quadratic equation is an equation which contains only one variable but that variable must only be raised to powers that are positive whole numbers with a maximum value of 2.

For example:	$x^2 + 2x + 1 = 0$	is a quadratic equation
	$2x^2 = 4$	is a quadratic equation
	$x^2 + \dfrac{1}{x} = 1$	is not a quadratic equation
	$x^3 + 2x^2 + 1 = 0$	is not a quadratic equation

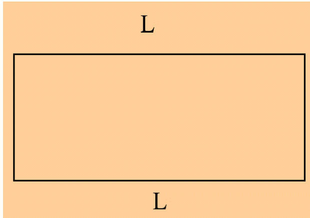
Quadratic functions can be represented graphically by a curve called a parabola. (See section 3.2.2)

Many problems that arise can be reduced to a quadratic equation.

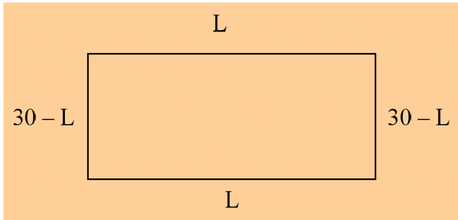
For example:

You have 60 m of fencing which you want to use to enclose a rectangular garden. Find an equation which will give you the area of the garden in terms of its length.

First draw the diagram and let L be the length of the garden.



If you have 60 m of fencing (and you will use it all) then the total amount used for the length is 2L and the total amount used for the width is 60 – 2L, so the width of the enclosed area must be 30 – L.



(To check, add the lengths of all the sides, this should be 60 m).

We know that area equals the product of length and width, so if area is A, then

$$A = L(30 - L)$$

$$A = 30L - L^2$$

(a quadratic formula).

If we knew the area we wanted, possibly $A = 100\ \text{m}^2$, then $100 = 30L - L^2$ would be a quadratic equation.

There are two techniques available to solve quadratic equations:

i. factorisation (see section 2.2.3)

ii. quadratic formula

i. Factorisation

Factorisation works on the principle that if the product of two expressions is zero then one or both of those expressions must be zero.

i.e. if

$$(x-a)(x-b)=0$$

then

$$x-a=0 \text{ or } x-b=0$$

and

$$x=a,\ x=b$$

Using techniques of factorisation in section 2.2.3 we know that

$$x^2+5x+6=(x+3)\,(x+2)$$

If

$$x^2+5x+6=0$$

then

$$(x+3)\,(x+2)=0$$

$$x+3=0 \text{ or } x+2=0$$

$$x=-3 \text{ or } x=-2$$

(Check answer by substituting back into original equation).

Exercise 2.12

Solve the quadratic equations:

a. $x^2+3x+2=0$

b. $x^2-5x+6=0$

c. $x^2-x-2=0$

d. $6x^2+7x+2=0$

e. $x^2-4=0$

Solutions

a. $x^2+3x+2=0$
$(x+1)(x+2)=0$
$\therefore x=-1 \text{ or } x=-2$

b. $x^2-5x+6=0$
$(x-3)(x-2)=0$
$\therefore x=3 \text{ or } x=2$

c. $x^2-x-2=0$
$(x-2)(x+1)=0$
$\therefore x=2 \text{ or } x=-1$

d. $6x^2+7x+2=0$
$6x^2+4x+3x+2=0$
$2x(3x+2)+(3x+2)=0$
$(3x+2)+(2x+1)=0$
$3x+2=0 \text{ or } 2x+1=0$
$\therefore x=\dfrac{-2}{3} \text{ or } x=\dfrac{-1}{2}$

e. $x^2-4=0$
$(x+2)(x-2)=0$
$\therefore x=-2 \text{ or } x=2$

ii. Quadratic formula

If the quadratic $ax^2 + bx + c = 0$ cannot be factorised (or if you cannot readily determine its factors), the solutions are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The + sign on the square root gives one solution and the – sign on the square root gives the other.

This formula can be derived as follows:

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{divide both sides by } a$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{subtract } \frac{c}{a} \text{ from both sides}$$

$$x^2 + 2\frac{b}{2a}x = -\frac{c}{a} \qquad \text{rewrite the coefficient of } x$$

$$x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{add } \left(\frac{b}{2a}\right)^2 \text{ to both sides to complete the square on LHS}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} \qquad \text{rewrite LHS}$$

(Completing the square technique is detailed in section 3.2.5)

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2} \qquad \text{put terms on RHS on common denominator}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \qquad \text{take square root of each side}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{subtract } \frac{b}{2a} \text{ from both sides}$$

Combining fractions on RHS yields

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula may be used to obtain the solutions of any quadratic equation.

Exercise 2.13

Solve:

a. $3x^2 - 15x + 17 = 9$ b. $3 = x + 4x^2$

Solutions

a. $3x^2 - 15x + 17 - 9 = 0$

$\therefore 3x^2 - 15x + 8 = 0$

Comparing with the standard form $ax^2 + bx + c = 0$
we note that $a = 3$, $b = -15$, $c = 8$

$$\therefore \ x = \frac{-(-15) \pm \sqrt{(-15)^2 - 4 \times 3 \times 8}}{2 \times 3} = \frac{15 \pm \sqrt{225 - 96}}{6}$$

$$= \frac{15 \pm \sqrt{129}}{6}$$

i.e. $x = \dfrac{15 + \sqrt{129}}{6}$ or $x = \dfrac{15 - \sqrt{129}}{6}$

$\therefore \ x \approx 4.39$ or $x \approx 0.61$

Check that each value of x satisfies the original equation

b. $4x^2 + x - 3 = 0$

Comparing with the standard form $ax^2 + bx + c = 0$
we note that $a = 4$, $b = 1$, $c = -3$

$$\therefore \ x = \frac{-1 \pm \sqrt{1 - 4 \times 4 \times (-3)}}{8} = \frac{-1 \pm \sqrt{49}}{8} = \frac{-1 \pm 7}{8}$$

$$= \frac{6}{8} \text{ or } \frac{-8}{8} = 0.75 \text{ or } -1$$

Check that each value of x satisfies the original equation

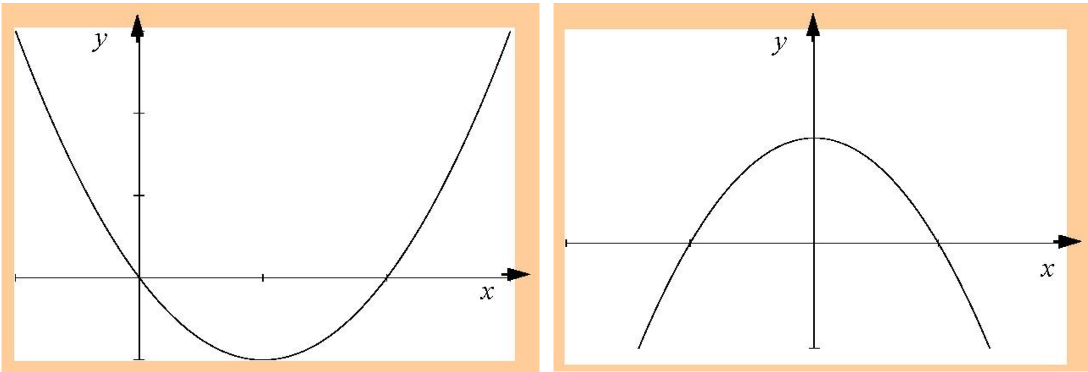
Types of solutions of quadratic equations

a. Two different solutions

The general equation of a parabola is a quadratic equation of the form

$y = ax^2 + bx + c$ (see section 3.2.3)

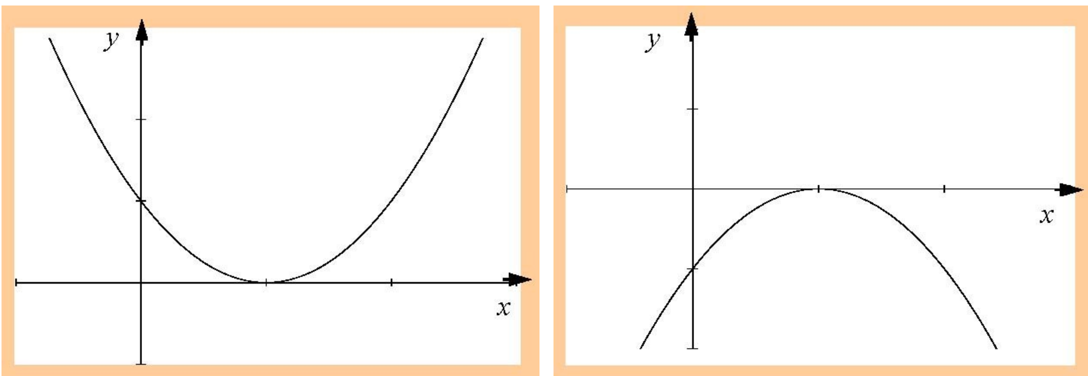
The solution of the equation when $y = 0$ (i.e. when we are considering the quadratic equation $ax^2 + bx + c = 0$) are the points where the curve cuts the x–axis. If there are two different solutions, the graph must be one of the two types below.



In equations of this type, the term $b^2 - 4ac$ in the quadratic formula is always positive.

b. One solution

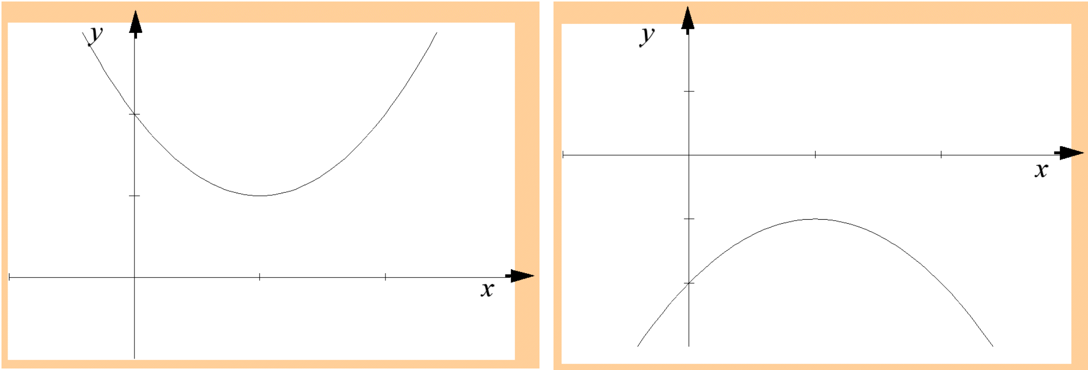
Graphs of this type must cut the x–axis at two equal points, i.e. the curve just touches the axis without crossing it.



In equations of this type, the term $b^2 - 4ac$ in the quadratic formula is zero.

c. No solution

If the graph is entirely above or entirely below the x–axis, then it does not touch (or cross) the x–axis at all.



These graphs lead to a negative value of $b^2 - 4ac$ in the quadratic formula. Since we cannot find the square root of a negative number, the equation has no real solution.

Note that the **type** of solution can be found by considering only the sign of $b^2 - 4ac$. No other calculation is required. Thus, if $b^2 - 4ac$ is

a. **positive**, two different solutions exist, i.e. there are 2 distinct values of x which satisfy the equation.

b. **zero**, one solution exists, i.e. there is only 1 value of x which satisfies the equation.

c. **negative**, no solution exists, i.e. there is no real value of x which satisfies the equation.

Examples

Example

Find the number of solutions of the quadratics:

a. $3x^2 - 4x + 1 = 0$ b. $x^2 - 6x + 9 = 0$ c. $2x^2 - 4x + 3 = 0$

Solutions

a. $b^2 - 4ac = 16 - 4 \times 3 \times 1 = 4.$ Thus, there are 2 solutions.

b. $b^2 - 4ac = 36 - 4 \times 9 = 0.$ Thus, there is 1 solution.

c. $b^2 - 4ac = 16 - 4 \times 2 \times 3 = -8.$ Thus, there is no real solution.

Example

A piece of sheet metal consists of a rectangle of length 2.43 metres and width $2r$ metres with a semi-circle of radius r metres on one end. If the total area is to be 15 m^2, find the value of r.

Solutions

The first task with problems like this is to develop an equation. Always draw a diagram if possible.

$$\text{Area of rectangle} = 4.86r$$

$$\text{Area of semicircle} = \frac{\pi r^2}{2}$$

$$\therefore \frac{\pi r^2}{2} + 4.86r = 15$$

$$\text{i.e. } \frac{1}{2}\pi r^2 + 4.86r - 15 = 0$$

This is a quadratic with $a = \frac{1}{2}\pi$, $b = 4.86$ and $c = -15$.

$$\therefore r = \frac{-4.86 \pm \sqrt{4.86^2 - 4 \times \frac{1}{2}\pi \times (-15)}}{2 \times \frac{1}{2}\pi}$$

$$= \frac{-4.86 \pm \sqrt{23.6169 + 30\pi}}{\pi} = \frac{-4.86 \pm \sqrt{117.87}}{3.142}$$

$$= \frac{-4.86 \pm 10.86}{3.142}$$

$$= 1.908 \quad \text{or} \quad -5.003$$

A negative value for the radius is impossible hence $r = 1.908$ m.

Exercise 2.14

Solve the following quadratic equations where possible.

a. $x^2 - 5x + 6 = 0$

b. $3x^2 - 2x + 4 = 0$

c. $x^2 - 5x + 2 = 0$

d. $2x^2 - 4x + 1 = 0$

e. $y^2 - 5y - 36 = 0$

f. $6t^2 + 17t = 3$

g. $x^2 - 8x + 9 = 0$

h. $2x^2 + 2x - 30 = 0$

i. $x^2 - 1.7x + 0.3 = 0$

j. $x^2 + 3.1x = 1.8$

k. $4x = x^2 - 45$

l. $m^2 = 14m - 49$

Solutions

a. $x = 2$ or 3

b. No real solution

c. $x = \dfrac{5 \pm \sqrt{17}}{2} = 4.56$ or 0.44

d. $x = 1.71$ or 0.29

e. $y = -4$ or 9

f. $t = -3$ or $\dfrac{1}{6}$

g. $x = 1.35$ or 6.65

h. $x = -4.41$ or 3.41

i. $x = 0.2$ or 1.5

j. $x = -3.6$ or 0.5

k. $x = -5$ or 9

l. $m = 7$

2.4.4 Polynomial equations

So far we have solved linear and quadratic equations. These are both examples of polynomial equations.

$y = 2$	constant equation
$y = mx + c$	linear equation
$y = ax^2 + bx + c$	the quadratic equation

This pattern can be extended to

$y = ax^3 + bx^3 + cx + d$	cubic equation etc.

The characteristic that makes these equations polynomial equations is that **all the powers of *x* are positive whole numbers or zero**. The degree of a polynomial equation is the value of its highest power.

Although not all polynomial equations will have a solution (or root) that is a real number, many of them can be solved by using the principles of factorisation. For example, if a cubic polynomial can be written as a product of three simple factors, then we could find its solution, i.e.

$$x^3 - 4x^2 + x + 6 = (x + 1)(x - 2)(x - 3) = 0$$

so

$x = -1$, $x = 2$ or $x = 3$ are solutions to this polynomial equation.

The difficulty lies in finding what the factors are. Two techniques are employed:

- inspection or trial and error
- division of polynomials

Inspection (or trial and error) is the best technique in the first instance, all solutions could be determined by guessing and checking.

For example, consider $x^3 - 4x^2 + x + 6 = 0$

$x = -1$ is a solution to the equation, i.e.

$$(-1)^3 - 4(-1)^2 + (-1) + 6 = 0$$

so we know that $(x + 1)$ must be a factor. Similarly, $x = 2$ and $x = 3$ would also satisfy the equation. We have just tried some possibilities to see if they work (usually we would try factors of the constant, 6, in the first instance).

An alternative method is to try inspection and division of polynomials.

Examples

Solve $x^3 - 4x^2 + x + 6 = 0$

Find one solution by guessing and checking.

Try $x = -1$, $(-1)^3 - 4(-1)^2 + (-1) + 6 = 0$.

So $x = -1$ is a solution and $x + 1$ must be a factor.

To find other factors divide $x + 1$ into $x^3 - 4x^2 + x + 6$ (for example, if we knew 8 was a factor of 24, we would find the other factors by dividing 24 by 8).

Using the technique of long division

$$
\begin{array}{rcl}
x^2 - 5x + 6 & & \\
x+1\,\overline{\big)\,x^3 - 4x^2 + x + 6} & \leftarrow & \text{Divide } x^3 \text{ by } x \text{ to get } x^2 \\
\underline{x^3 + x^2} \qquad\qquad & \leftarrow & \text{Multiply } x + 1 \text{ by } x^2 \\
-5x^2 + x + 6 & \leftarrow & \text{Subtract } x^3 + x^2 \text{ from entire polynomial, term by term} \\
 & & \text{Divide } -5x^2 \text{ by } x \text{ to get } -5x \\
\underline{-5x^2 - 5x} \qquad & \leftarrow & \text{Multiply } x + 1 \text{ by } -5x \\
6x + 6 & \leftarrow & \text{Subtract } -5x^2 - 5x \text{ from remaining polynomial} \\
\underline{6x + 6} & \leftarrow & \text{Divide } 6x \text{ by } x \text{ to get } 6 \\
 & & \text{Multiply } x + 1 \text{ by } 6 \\
0 & & \text{Subtract } 6x + 6 \text{ from remaining polynomial}
\end{array}
$$

Thus

$$
\begin{aligned}
x^3 - 4x^2 + x + 6 &= (x + 1)(x^2 - 5x + 6) \quad \text{Factorise using standard techniques} \\
&= (x + 1)(x - 3)(x - 2)
\end{aligned}
$$

So solutions to the polynomial equation are

$$x = -1,\ x = 3, \text{ or } x = 2$$

Check answer by substituting back into the original equation.

Example

Solve $x^3 + 2x^2 - x - 2 = 0$

Try solutions $x = \pm 1$ and $x = \pm 2$

$x = -1$, $(-1)^3 + 2(-1)^2 - (-1) - 2 = 0$

So $x = -1$ is a solution and $x + 1$ is a factor.

To find other factors use long division:

$$
\begin{array}{rllll}
 & & x^2 & + x & - 2 \\ \cline{2-5}
x + 1 \big) & x^3 & + 2x^2 & - x & - 2 \\
 & \underline{x^3} & \underline{+ x^2} & & \\
 & & x^2 & - x & - 2 \\
 & & \underline{x^2} & \underline{- x} & \\
 & & & -2x & - 2 \\
 & & & \underline{-2x} & \underline{- 2} \\
 & & & & 0
\end{array}
$$

Thus $(x + 1)(x^2 + x - 2)$ are factors.

Factorising the quadratic:

$$(x + 1)(x + 2)(x - 1) = 0$$

Solutions are $x = -1$, $x = -2$ or $x = 1$

Check solutions by substituting all solutions back into the original polynomial equation.

Example

Solve $2x^3 - 3x^2 - 29x - 30 = 0$

Try ±2, ±1, ±5, ±6

If $x = -2$, $2(-2)^3 - 3(-2)^2 - 29(-2) - 30 = 0$

So $x = -2$ is a solution and $x + 2$ is a factor.

To find the other factors use long division:

$$
\begin{array}{r l}
 & \phantom{2x^3 - {}}2x^2 - 7x - 15 \\ \cline{2-2}
x + 2 \big) & 2x^3 - 3x^2 - 29x - 30 \\
 & \underline{2x^3 + 4x^2} \\
 & - 7x^2 - 29x - 30 \\
 & \underline{- 7x^2 - 14x} \\
 & - 15x - 30 \\
 & \underline{- 15x - 30} \\
 & 0
\end{array}
$$

Thus $(x + 2)\ (2x^2 - 7x - 15)$ are factors.

Factorising the quadratic we get

$(x + 2)(2x + 3)(x - 5) = 0$

So solutions are $x = -2$, $x = -\frac{3}{2}$, or $x = 5$

Check solutions by substituting back into the original polynomial equation.

Exercise 2.15

Solve the following polynomial equations for x.

a. $x^3 - 19x + 30 = 0$

b. $6x^3 - 5x^2 - 2x + 1 = 0$

c. $x^3 + 2x^2 - x - 2 = 0$

d. $5x^3 + 12x^2 - 36x - 16 = 0$

Solutions

a. $x = 2, x = 3$ or $x = -5$

b. $x = 1, x = \frac{1}{3}$ or $x = -\frac{1}{2}$

c. $x = 1, x = -1$ or $x = -2$

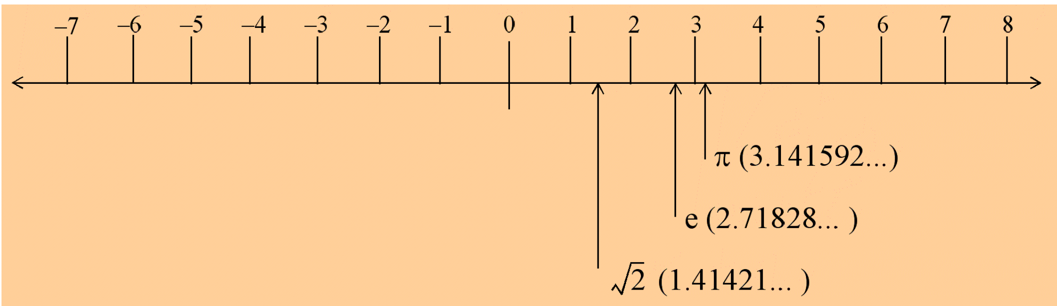
d. $x = 2, x = -\frac{2}{5}$ or $x = -4$

2.4.5 Inequalities

Real numbers are made up of rational numbers (integers and fractions) and irrational numbers (those numbers which can be represented by a decimal which does not terminate or repeat), e.g.

$\sqrt{2}, e, \pi$

These can be displayed on a line called the real number line as follows:



An **inequality** (sometimes called an inequation) is a relationship which holds between two numbers (or algebraic expressions) which are not equal.

If a and b are any two real numbers and are represented by points on the number line directed to the right, then $a > b$ if and only if the graph of a lies to the right of the graph of b (and $a < b$ if and only if the graph of a lies to the left of the graph of b).

For example, we know that the number 3 is less than the number 5 and we can write this as the inequality $3 < 5$ where '<' is the symbol for 'less than'. We could also write this inequality as $5 > 3$ where '>' is the symbol for 'greater than'.

Note that inequalities are usually read from left to right so $2 > 1$ is read as ' two is greater than one' and $4 < 6$ is read as ' four is less than six' . The inequality $x > 2$ is read as 'x is (any real number) greater than two' . Alternatively we could use interval notation and write that the range of x is $(2,\infty)$. Some valid values of x are 2.1, 3, 125, 5 etc. The inequality $y < 10$ is read as 'y is (any real number) less than 10' or the range of y is $(-\infty, 10)$. Some valid values of y are 9.9, 8, 7.64, 2, –127.65 etc.

If we wished to allow y to take on the value of 10 as well as any real number below 10, we can write $y \le 10$ where '≤' is the symbol ' less than or equal to' or the range of y is $(-\infty,10]$. Note a square bracket is used to indicate that the end point of the interval is also a valid value of y. Infinity end points always have round brackets (because infinity is not a number). Similarly we can write $x \ge 2$ meaning x is greater than or equal to 2 or the range of x is $[2,\infty)$.

The range of values that x can take can be further restricted by using two inequality symbols.

e.g. $1 \le x \le 3$ is read as 'x is greater than or equal to one but less than or equal to three' or $x = [1, 3]$.

Other possibilities are:

$1 < x \leq 3$ $\quad$ x is greater than 1 but less than or equal to 3 or $x = (1, 3]$

$1 < x < 3$ $\quad$ x is greater than 1 but less than 3 or $x = (1, 3)$

$1 \leq x < 3$ $\quad$ x is greater than or equal to one but less than 3 or $x = [1, 3)$

Note that inequalities like $1 > x > 3$ do not make sense since they imply that 1 is greater than 3 which of course is not the case. Do not use inequalities like $2 > x < 4$. Use two separate inequalities $x < 2$, $x < 4$, and if x is to satisfy both of these, then x must be less than two, i.e. the second inequality is superfluous and all we need is the inequality $x < 2$ or $x = (-\infty, 2)$.

Examples

Express the following inequalities in interval notation.

a. $x > -4$ $\quad$ b. $x \leq 32$ $\quad$ c. $-10 < x \leq -2.1$

d. $x \geq 4$ $\quad$ e. $4 \leq x \leq 8$

Solutions

a. $(-4, +\infty)$ $\quad$ b. $(-\infty, 32]$ $\quad$ c. $(-10, -2.1]$

d. $[4, +\infty)$ $\quad$ e. $[4, 8]$

So far we have discussed inequalities only with a single term in the unknown x.

Inequalities involving expressions of an unknown play a significant role in many areas of mathematics, science and engineering and it is important that you are able to understand and manipulate them.

Consider the inequality $3x^2 - 2x \leq 4x^2 - 3x + 2$.

We can use the same rules for manipulating inequalities as those used for transposing equations except that **if the inequality is multiplied or divided by a negative number**, **the direction of the inequality is reversed**.

To illustrate the importance of this change in the direction of the inequality, consider the inequality $3 > 2$.

If both sides are multiplied by -1 and the direction of the inequality is not reversed, we have $-3 > -2$ which of course is wrong. The correct answer is $-3 < -2$.

To simplify the above inequality we have

	$3x^2 - 2x \le 4x^2 - 3x + 2$	
$\therefore$	$3x^2 - 4x^2 - 2x \le -3x + 2$	Subtracting $4x^2$ from both sides
$\therefore$	$-x^2 - 2x \le -3x + 2$	
$\therefore$	$-x^2 - 2x + 3x \le 2$	Adding $3x$ to both sides
$\therefore$	$-x^2 + x \le 2$	
or	$x^2 - x \ge -2$	Multiplying both sides by –1 and changing the direction of the inequality

Examples

Simplify the following inequalities

a. $4x - 8 < 9$

b. $2x + 4 > 7x - 9$

c. $2x + 2 \le 4x - 3$

d. $2x^2 + 4x \ge 3x^2 - 4x + 9$

e. $|x| < 8$

f. $|3x + 2| < 8$

g. $|x - 2| \ge -6$

h. $|x + 3| \ge 5$

Solutions

a. $4x - 8 < 9 \quad \therefore 4x < 17$

$$\therefore\ x < \frac{17}{4} \qquad \therefore\ x = \left(-\infty, \frac{17}{4}\right)$$

b. $2x + 4 > 7x - 9 \quad \therefore 2x - 7x > -9 - 4 \quad \therefore -5x > -13$

$\therefore\ x = < \dfrac{13}{5}$ Dividing by –5 changes the direction of the inequality

or $x = \left(-\infty, \dfrac{13}{5}\right)$

c. $2x + 2 \le 4x - 3 \quad \therefore 2x - 4x \le -3 - 2 \quad \therefore -2x \le -5$

$\therefore\ x = \ge \dfrac{5}{2}$ Dividing by –2 changes the direction of the inequality

or $x = \left[\dfrac{5}{2}, +\infty\right)$

d. $2x^2 + 4x \ge 3x^2 - 4x + 9 \quad \therefore 2x^2 - 3x^2 + 4x + 4x \ge 9 \quad \therefore -x^2 + 8x \ge 9$

e. Recall that the symbol $|x|$ means the absolute value of x, or the magnitude of x.

i.e. $|x| = \begin{cases} x, x \geq 0 \\ -x, x < 0. \end{cases}$

For example

$|-3| = -(-3) = 3$, $|5| = 5$.

Hence, for $|x| < 8$, there are two cases to consider:

case 1. $x \geq 0$

We then have $|x| = x$ and so $x < 8$.

Putting this together gives $0 \leq x < 8$.

case 2. $x < 0$

We then have $|x| = -x$ and so $-x < 8$, or $x > -8$. Putting this together gives $-8 < x < 0$.

Now we combine case 1 and case 2 to get $-8 < x < 8$.

Note that in general if $|x| < a$ where a is positive, then $-a < x < a$. This is a most useful thing to remember.

f. $|3x + 2| < 8$

Since $|x| <$ a means that $-a < x < a$, then we have $-8 < 3x + 2 < 8$, and so subtracting 2 right through gives $-10 < 3x < 6$.

Divide by 3 to get $\frac{-10}{3} < x < 2.$

g. $|x - 2| \geq -6$

Any value of x satisfies this inequality, since the absolute value is always positive, and so is greater than or equal to –6. Hence the answer is all x.

h. $|x + 3| \geq 5$

Now either $|x + 3| = x + 3$ if $x \geq -3$ or $|x + 3| = -(x + 3)$ if $x < -3$ (see definition of absolute value in (e) above).

Hence

Either	$x + 3 \geq 5$	or	$-(x + 3) \geq 5,$	
	$x \geq 2$	or	$x + 3 \leq -5,$	Multiply by –1 changes the inequality
	$x \geq 2$	or	$x \leq -8.$	

Hence either x is less than or equal to –8 or x is greater than or equal to 2 (obviously it cannot satisfy both inequalities at the same time).

Exercise 2.16

a. Express the following in interval notation

i. $x < 3$

ii. $x \geq -3$

iii $-7 < x \leq -2$

iv. $-8.24 \leq x \leq 10$

b. Express the following as inequalities

i. $x = [100,101]$

ii. $x = (-\infty, 4)$

iii. $x = [-4, +\infty)$

c. Simplify the following inequalities

i. $10x + 2 \leq 18x - 4$

ii. $3x^3 - 2x^2 > -3x^2 + x - 4$

iii $|2x - 2| > 6$

iv. $|2x - 2| < 6$

Solution

a.

i. $(-\infty, 3)$

ii. $[-3, +\infty)$

iii $(-7, -2]$

iv. $[-8.24, 10]$

b.

i. $100 \leq x \leq 101$

ii. $x < 4$

iii. $x \geq -4$

c.

i. $x \geq \frac{3}{4}$

ii. $3x^3 + x^2 - x > -4$

iii $x > 4$ **or** $x < -2$

iv. $-2 < x < 4$

Module 2: Self assessment

Questions 2.1

1. Simplify $12 + 2x^2 + 4x^{-1} + 2 - 6x^{-1} + x^2$
2. Expand $(2x + 1)(x - 3)(x + 2)$
3. Factorise $2x^2 + 3x - 2$
4. Make t the subject of the formula $L = \dfrac{1}{2\pi r}\sqrt{\dfrac{t}{4} + 1}$
5. Solve for x: $3x^2 - 2x - 2 = 0$
6. Solve for x: $x^3 - 2x^2 - x + 2 = 0$
7. Solve for x: $|3x - 1| < 6$
8. Write as a single fraction with the lowest possible denominator. $\dfrac{x+1}{x+2} - \dfrac{x-1}{x-3}$
9. Solve for x: $2x + \dfrac{1}{3} = -\dfrac{4}{3}x - 1$

Questions 2.2

1. Simplify $\frac{b}{3} + a^2 - \frac{2b^2}{7} + \frac{3}{2}a^2 + \frac{5}{2}b^2$
2. Expand $-(x-3)(2x+1)(x-3)$
3. Factorise $6x^2 + x - 1$
4. Make y the subject of the formula $\frac{1}{x} = \frac{1}{a} + \frac{1}{y}$
5. Solve for x: $-3x^2 + 3x + 1 = 0$
6. Solve for x: $x^3 - 2x^2 - 5x + 6 = 0$
7. Solve for x: $-|x+2| < -2$
8. Write as a single fraction with the lowest possible common denominator:
   $-\frac{(x+1)}{2x-1} - \frac{x}{x-1}$
9. Solve for x: $-\frac{1}{2} - \frac{x}{2} = 2x - \frac{1}{3}$

Module 3 – Functions and relations

Objectives

In this module you are required to be able to:

- demonstrate understanding of the nature of functions
- apply functional notation
- recognise equations and graphs of straight lines, parabolas, cubics, hyperbolas and circles
- solve the intersection of functions algebraically and graphically
- sketch graphs of straight lines, parabolas, cubics, hyperbolas and circles.

3.1 What is a function?

In the previous sections we were concerned with variables . . . . variables in expressions and variables in equations. However, the study of mathematics is concerned with the way in which different variables are related to each other. This relationship can be described both algebraically and graphically.

For example, consider what happens to a length of wire when weights are added to it. You might expect (especially if you are familiar with Hooke's Law) that the heavier the weight, the more the wire stretches. In fact we get the following results:

Weight, *w*. (newtons)	**Length, *l*. (metres)**
20	2.00042
40	2.00086
60	2.00182
80	2.00176
100	2.00219

Notice that at any given weight we only have one length. We call weight the independent variable and length the dependent variable, and say that length is dependent on weight or **length is a function of weight**.

We could think of any function as a set of such ordered pairs in which each first member has exactly one second member.

A special notation has been developed to use with functions. For example

We say: length is a function of weight

We write: $l = f(w)$

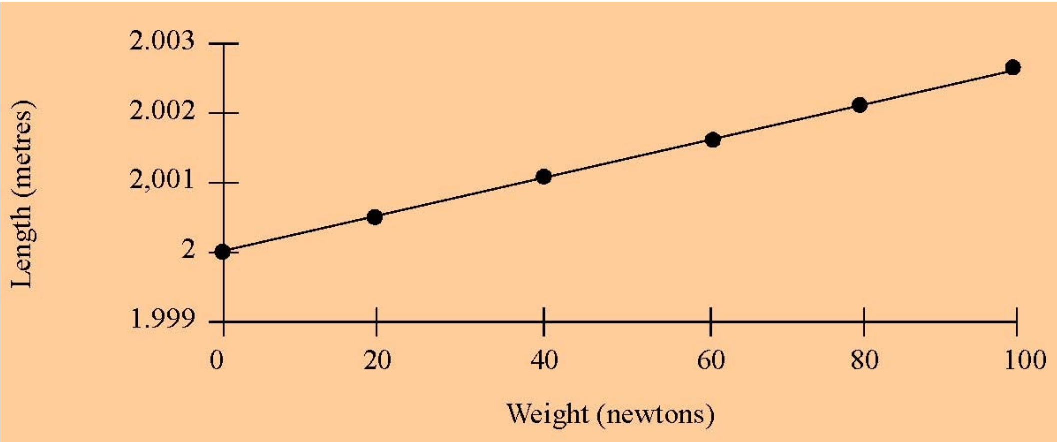
We may in fact take this one step further and generate from the data a formula to represent the data. In the stretching wire case this would be

$$l = 0.000023w + 2$$

or alternatively using function notation

$$l = f(w) = 0.000023w + 2$$

Alternatively we could use the data to draw a graph of the relationship.



Graph showing length against weight for a steel wire

This of course is a straight line or linear function. We will discuss the characteristics of linear functions later in this material.

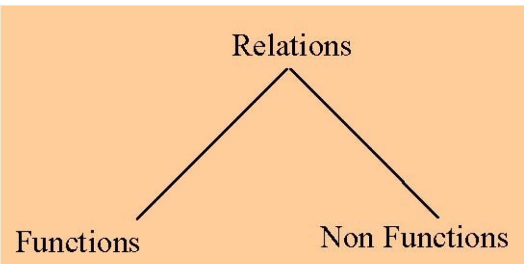
For functions in general we need to know some specific characteristics and language.

A function is a set of ordered pairs in which for each first member there is exactly one second member. Functions can be represented by equations relating the two variables.

The **domain** of a function is the set of first members of the ordered pairs (often denoted by x).

The **range** of a function is the set of second members of the ordered pairs (often denoted by y).

A relation is any set of ordered pairs, so that functions are a subset of relations.



It is easy to determine whether a relation is a function or not by looking at its graph. If a vertical line can cut the graph in exactly one place everywhere in the graph then the relation is a function.

We will examine specific functions and relations with their associated domains and ranges later in this section.

To investigate the concept of the function further you might want to view the video 'The F Files' located in USQ library.

3.1.1 Algebra of functions

The most commonly used notation of a function is of the form $f(x)$ (read 'f of x'). However, we are free to use any letters to denote a function . . . . others that are often used are g and h.

How do we do calculations with functions?

Examples

Example

If $f(x) = x^2 - 1$, find $f(2)$, $f(-3)$, $f(h)$.

This question means that we have to calculate the value of the function $f(x)$ at, $x = 2$, $x = -3$ and $x = h$.

i. To evaluate $f(2)$ if $f(x) = x^2 - 1$ then

$f(2) = 2^2 - 1$ Substitute $x = 2$ for every x

$f(2) = 3$

So $f(2) = 3$, or when $x = 2$, $f(x) = 3$.

ii. If $f(x) = x^2 - 1$ Substitute $x = -3$ for every x

$f(-3) = (-3)^2 - 1$

$f(-3) = 8$

iii. If $f(x) = x^2 - 1$ Substitute $x = h$ for every x

$f(h) = h^2 - 1$

Example

Calculate the value of the function $f(x) = x^2 - 4x + 5$ when $x = x + h$ (i.e. evaluate $f(x + h)$).

$f(x) = x^2 - 4x + 5$ Substitute $x = x + h$ for every x

$f(x + h) = (x + h)^2 - 4(x + h) + 5$

$f(x + h) = x^2 + 2xh + h^2 - 4x - 4h + 5$

$f(x + h) = x^2 + (2h - 4)x + h^2 - 4h + 5$

Example

If $f(x) = 2x + 1$ and $g(x) = 2x^2 - 3x$, calculate $f(g(x))$.

$f(g(x)) = 2g(x) + 1$ Substitute $g(x)$ for every x

$= 2(2x^2 - 3x) + 1$

$= 4x^2 - 6x + 1$

Exercise 3.1

a. If $f(x) = 3x - 1$, calculate

i. $f(0)$

ii. $f(-1)$

iii. $f\left(\frac{1}{2}\right)$

iv. $f(a)$

b. If $f(x) = 2x^2 - 3x + 1$, calculate

i. $f(-1)$

ii. $f(a+1)$

iii. $f(-a)$

iv. $f\left(\frac{1}{x}\right)$

c. If $f(x) = \frac{1}{2}x + 1$ and $g(x) = x^2 + x - 1$, calculate

i. $f(g(x))$

ii. $g(f(x))$

d. If $f(x) = 2x^2 - x + 1$, calculate

i. $f(h)$

ii. $f(x+h)$

iii. $\frac{f(x+h) - f(x)}{h}$ (Note this relationship is important in later calculus work)

Solutions

a. If $f(x) = 3x - 1$

i. $f(0) = 3 \times 0 - 1 = -1$

ii. $f(-1) = 3 \times -1 - 1 = -4$

iii. $f\left(\frac{1}{2}\right) = 3 \times \frac{1}{2} - 1 = \frac{3}{2} - 1 = \frac{1}{2}$

iv. $f(a) = 3a - 1$

b. If $f(x) = 2x^2 - 3x + 1$, calculate

i. $f(-1) = 2(-1)^2 - 3 \times -1 + 1 = 2 + 3 + 1 = 6$

ii.
$$\begin{aligned} f(a+1) &= 2(a+1)^2 - 3(a+1) + 1 \\ &= 2(a^2 + 2a + 1) - 3a - 3 + 1 \\ &= 2a^2 + 4a + 2 - 3a - 3 + 1 \\ &= 2a^2 + a \end{aligned}$$

iii.
$$\begin{aligned} f(-a) &= 2(-a)^2 - 3(-a) + 1 \\ &= 2a^2 + 3a + 1 \end{aligned}$$

iv.
$$\begin{aligned} f\left(\frac{1}{x}\right) &= 2\left(\frac{1}{x}\right)^2 - 3\left(\frac{1}{x}\right) + 1 \\ &= \frac{2}{x^2} - \frac{3}{x} + 1 \\ &= \frac{2 - 3x + x^2}{x^2} \\ &= \frac{x^2 - 3x + 2}{x^2} \end{aligned}$$

c. If $f(x) = \frac{1}{2}x + 1$ and $g(x) = x^2 + x - 1$, calculate

i.
$$\begin{aligned} f(g(x)) &= \frac{1}{2}g(x) + 1 \\ &= \frac{1}{2}\left(x^2 + x - 1\right) + 1 \\ &= \frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{2} + 1 \\ &= \frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{2} \\ &= \frac{1}{2}\left(x^2 + x + 1\right) \end{aligned}$$

ii. $$g\left(f\left(x\right)\right) = \left(f\left(x\right)\right)^2 + \left(f\left(x\right)\right) - 1$$
$$= \left(\frac{1}{2}x+1\right)^2 + \left(\frac{1}{2}x+1\right) - 1$$
$$= \frac{1}{4}x^2 + x + 1 + \frac{1}{2}x + 1 - 1$$
$$= \frac{1}{4}x^2 + \frac{3}{2}x + 1$$

d. If $f(x) = 2x^2 - x + 1$, calculate

i. $f(h) = 2h^2 - h + 1$

ii. $$f(x+h) = 2(x+h)^2 - (x+h) + 1$$
$$= 2(x^2 + 2xh + h^2) - x - h + 1$$
$$= 2x^2 + 4xh + 2h^2 - x - h + 1$$
$$= 2x^2 + (4h-1)x + 2h^2 - h + 1$$

iii. $$\frac{f\left(x+h\right) - f\left(x\right)}{h} = \frac{2x^2 + 4xh + 2h^2 - x - h + 1 - \left(2x^2 - x + 1\right)}{h}$$
$$= \frac{2x^2 + 4xh + 2h^2 - x - h + 1 - 2x^2 + x - 1}{h}$$
$$= \frac{4xh + 2h^2 - h}{h}$$
$$= \frac{h\left(4x + 2h - 1\right)}{h}$$
$$= 4x + 2h - 1$$

3.2 Graphs of functions and relations

To graph functions and relations we need to understand the system which makes that possible.

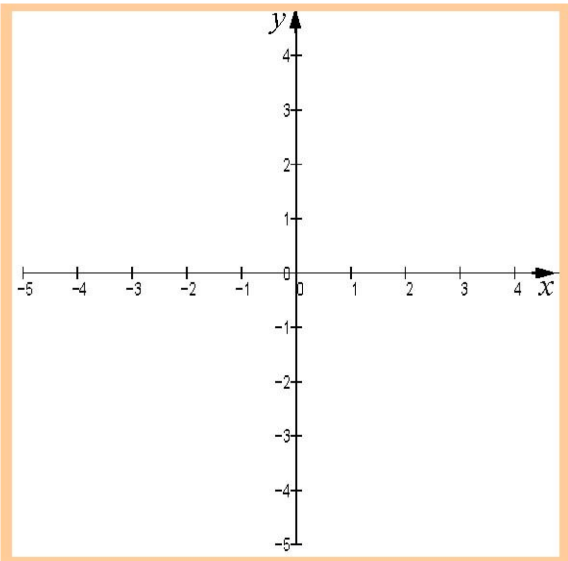
3.2.1 Cartesian coordinate system

There are many instances where it is necessary to refer to a point in a plane and to be able to distinguish that point from other points in the plane.

A system invented by the French mathematician and philosopher, René Descartes (1596–1650) makes this possible. This system is called the Cartesian Coordinate system. When applied to a plane, the plane may be called the Cartesian plane or the xy-plane.

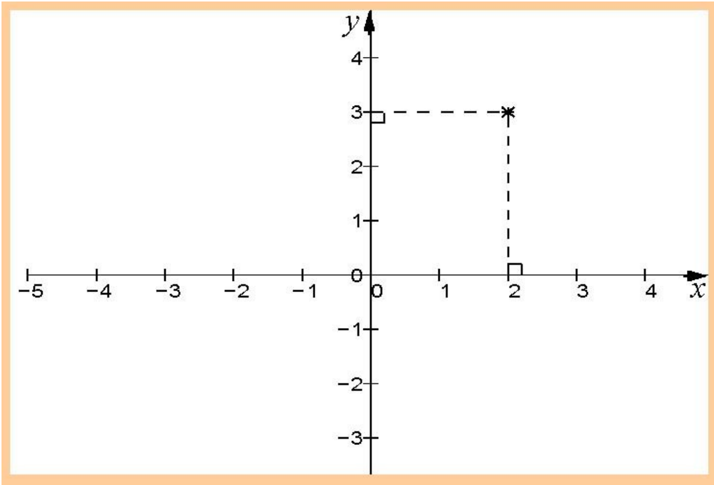
To construct a Cartesian coordinate system in a plane, two mutually perpendicular lines are drawn intersecting at a point called the origin (the zero point). Units of length, usually but not necessarily the same, are marked off on each of these perpendicular lines as in the figure below.

The perpendicular lines are coordinate lines or axes. The vertical coordinate line is the y-axis and the horizontal coordinate line is the x–axis.

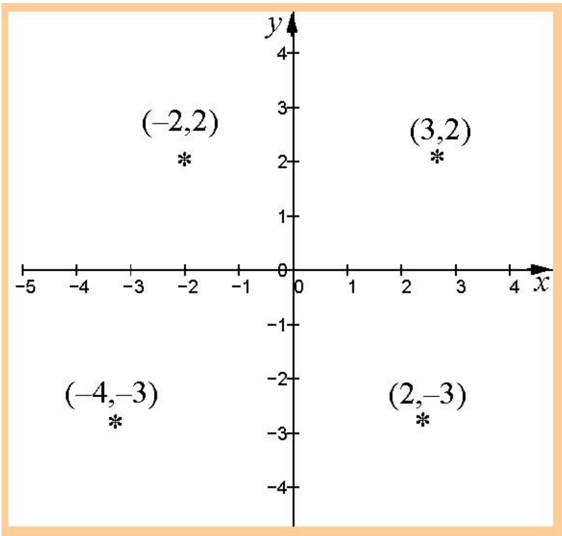


Any point P in the plane now has two numbers associated with it. Its x-coordinate (say x_1) and its y-coordinate (say y_1) are written as an ordered pair, (x_1,y_1) and called the coordinates of P.

To determine this ordered pair for a particular point P, drop a perpendicular from P to the x-axis and from P to the y-axis. The point of intersection of these perpendiculars and the x-axis and y-axis are respectively the x-coordinate and y-coordinate of the point P. In the figure below coordinates of P are (2,3). Alternatively we say P is the point (2,3).



The figure below gives the coordinates of various points in the xy-plane. Note that the general point in the Cartesian plane has coordinates (x,y) and the origin 0 is (0,0).



Many different functions and relations can be represented pictorially using this system. The following include the most commonly occurring functions and relations.

3.2.2 The linear function

In a previous section we examined how the length of a piece of wire varied when different weights were added to it. This relationship was a function of the form

$$l = f(w) = 0.000023w + 2$$

because for every weight there was a unique value for the length.

When this function was graphed on a coordinate system it was clear it was a straight line with a positive slope. In fact the line has a slope of 0.000023 and cuts the vertical axis at 2.

The form presented above is typical of a linear or straight line function.

In general straight line functions can be represented in a number of forms.

Typically, the equation of a straight line can be expressed as

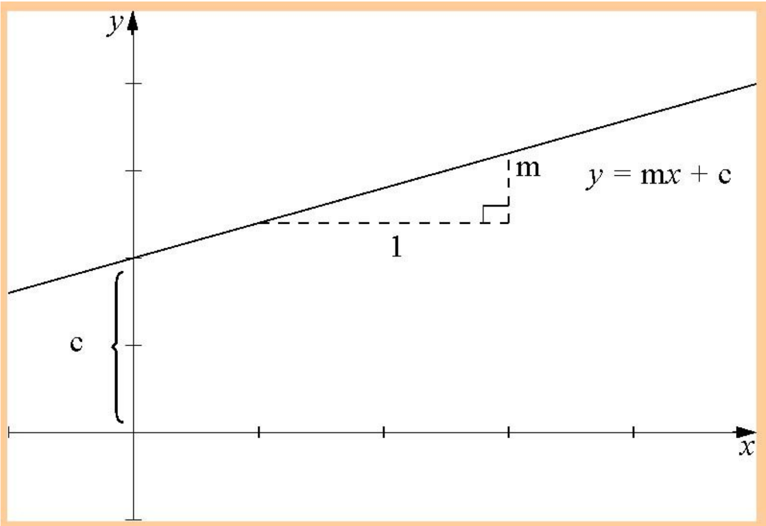
$$\boxed{y = \mathrm{m}x + \mathrm{c}}$$

Here c and m are constants.

c is the **intercept** on the y-axis, i.e. the value of y when $x = 0$.

m is the **slope** or gradient of the line, i.e. the change in y for a unit change in x.

$y = \mathrm{m}x + \mathrm{c}$ is the slope-intercept form of the equation of a straight line.



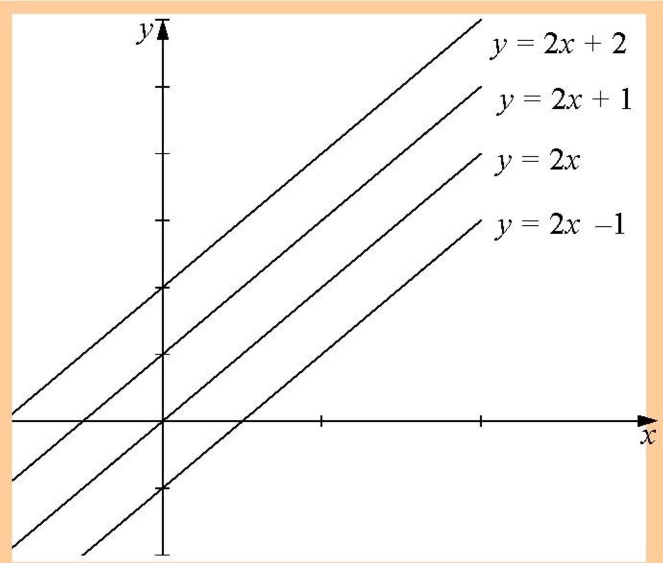
The slope and intercept of a straight line

All points (x,y) which satisfy the equation of the line lie on the line.

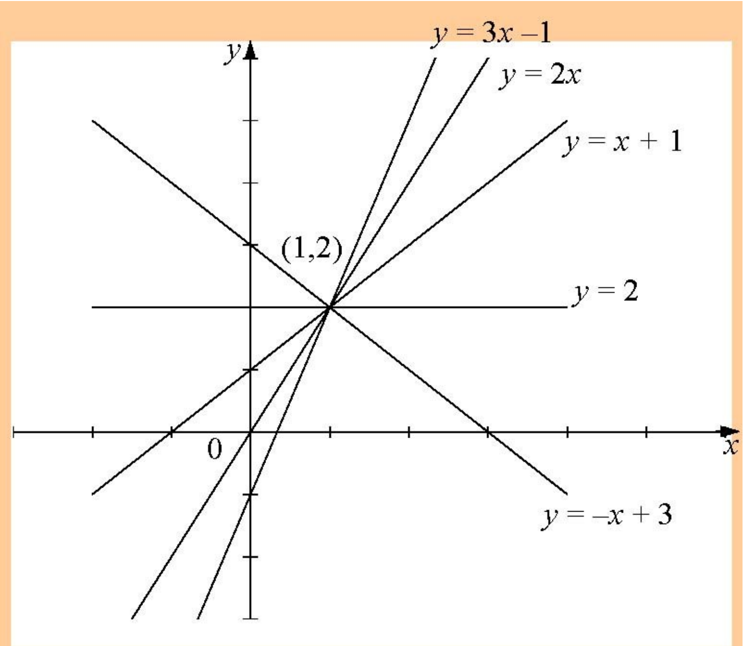
To draw a straight line find any two points that lie on the line and join them with a straight line.

Lines with the same slope are parallel lines. Lines with a point in common are called concurrent or intersecting lines. Lines that are both parallel and concurrent are **coincident**, i.e. their graphs lie on top of each other. There are infinite numbers of solutions to the system representing the equations of coincident lines as there are an infinite number of points common to the lines.

Note: Parallel lines never intersect so they do not have a point in common. Hence there is no solution to the system of equations representing the lines.



Parallel lines



Concurrent lines

Note: Intersecting lines have only one point in common. Hence there is a unique solution to the system of equations representing the lines.

Exercise 3.2

Find the slopes and intercepts of the lines below and sketch their graphs.

a. $y = 3x + 2$

b. $y = 3 - x$

c. $x = 10 - 2y$

d. $3x - 4y - 5 = 0$

Solutions

a. $y = 3x + 2$

y intercept is 2, slope is 3.

The easiest points to find to sketch the graph of a straight line are the points where the graph cuts the axes. We know the y intercept is 2.

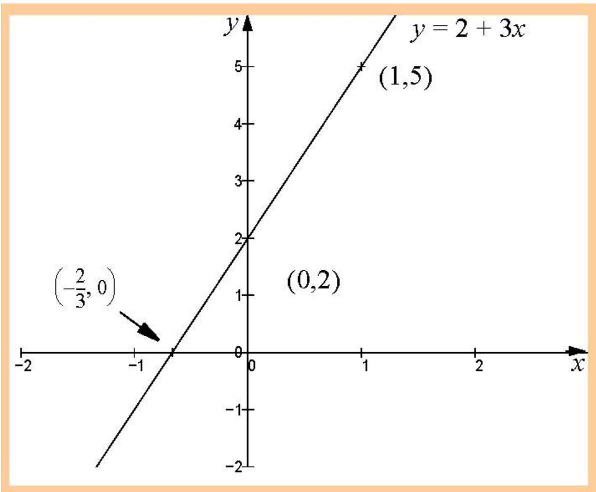
$\therefore$ One point is (0,2).

The point where the graph cuts the x-axis is where $y = 0$.

When $y = 0$, from $y = 2 + 3x$ we have $0 = 2 + 3x$.

$\therefore x = -\dfrac{2}{3}$ is the x-intercept and $\left(-\dfrac{2}{3}, 0\right)$ is another point on the graph.

Note: We could have used any value of x say $x = 1$ and found the corresponding y value (i.e. $y = 2 + 3 \times 1 = 5$) $\rightarrow$ (1,5) is another point on the graph.

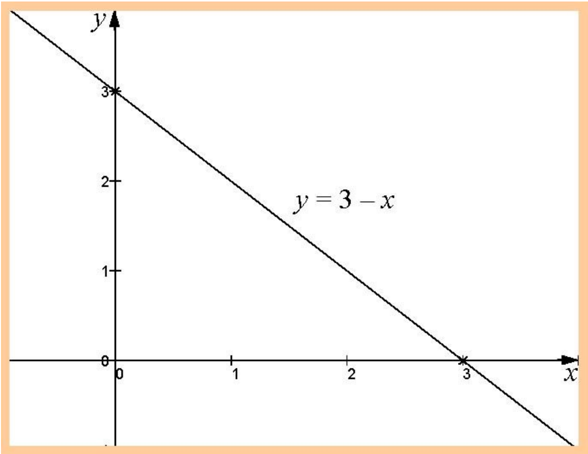


b. $y = 3 - x$

y intercept is 3, slope is -1.

y intercept is 3. $\therefore$ (0,3) is on the line and when $y = 0$, $0 = 3 - x$

$\therefore x = 3$ is the x-intercept and (3,0) is on the line

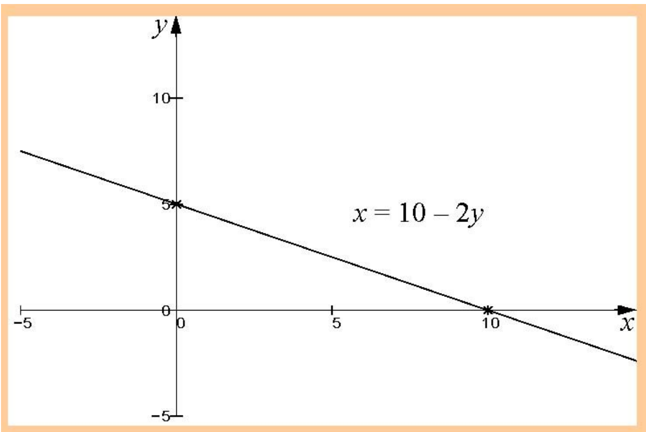


c. $x = 10 - 2y$

Rearranging gives $y = -\dfrac{1}{2}x + 5$ in slope-intercept form.

y intercept is 5, slope is $-\dfrac{1}{2}$

Two points on the line are (0,5) and (10,0).

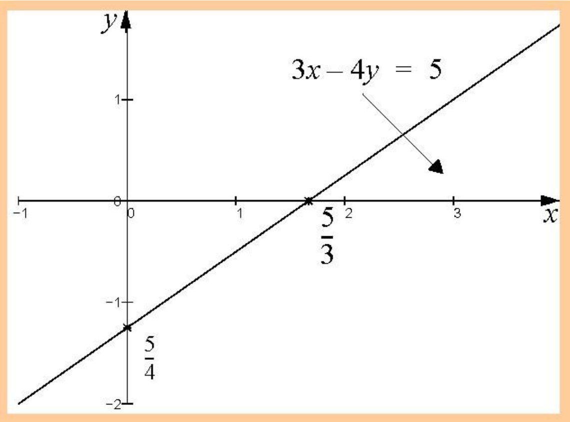


d. $3x - 4y = 5$

Rearranging gives $y = -\frac{3}{4}x - \frac{5}{4}$ in slope-intercept form

y intercept is $\frac{-5}{4}$, slope is $\frac{3}{4}$.

Two points on the line are $\left(0, \frac{-5}{4}\right)$ and $\left(\frac{5}{3}, 0\right)$



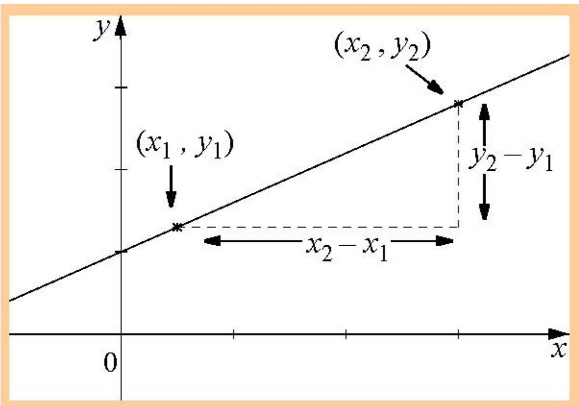
Slope-point equation

If the slope, m, of a line and one point, say (x_1,y_1), on the line are known, then the equation of the line may be found as follows:

If (x_2,y_2) is any other point on the line then the change in y for moving from (x_1,y_1) to (x_2,y_2) is $y_2 - y_1$ and the corresponding change in x is $x_2 - x_1$.

$\therefore$ The change in y for a unit change in x is $\text{m} = \frac{y_2 - y_1}{x_2 - x_1}$ i.e. $\text{m} = \frac{\text{rise}}{\text{run}}$

But (x_2,y_2) is any other point on the line so it may be replaced by (x,y) to obtain $\text{m} = \frac{y - y_1}{x - x_1}$



so $y - y_1 = \mathrm{m}(x - x_1)$, therefore

$$\boxed{y = y_1 + \mathrm{m}(x - x_1)}$$

which is the slope-point form of the equation of a straight line.

Exercise 3.3

Find the equations of the straight lines

a. passing through (1,–1) with slope 1

b. passing through (2,0) with slope $-\frac{1}{2}$

c. passing through (–3,1) with slope 2

Solutions

a. Here $x_1 = 1$, $y_1 = -1$, m = 1 so the required equation is

$$y = -1 + 1\,(x - 1)$$

$$\therefore y = -2 + x$$

b. Here $x_1 = 2$, $y_1 = 0$, m = $-\frac{1}{2}$ and the required equation is

$$y = 0 - \frac{1}{2}(x - 2)$$

$$\therefore\ y = 1 - \frac{1}{2}x$$

c. Here $x_1 = -3$, $y_1 = 1$, m = 2 and the required equation is

$$y = 1 + 2(x - (-3))$$

$$\therefore y = 7 + 2x$$

Note: You should always check your answer by substituting x_1 for x in the equation e.g. in c. put $x = x_1 = -3$ and we have $y = 7 + 2\,(-3) = 1 = y_1$ so we have the right equation.

Applications of linear functions

Examples

If you descend into the ocean, pressure increases linearly. At the surface the pressure is equal to atmospheric pressure (100 kilopascals (kPa)), while at a depth of 10 m (approx) the pressure is twice atmospheric pressure.

i. Write an expression for pressure as a function of depth

ii. What is the pressure at 8000 m depth?

iii. Graph the equation for the domain $0 \le d \le 8000$

Solutions

i. Let pressure in kPa be P, and depth in metres be d. We know from the question that at

$d = 0$, $P = 100$ and $d = 10$, $P = 200$

If we have the general equation for a straight line $y = \mathrm{m}x + \mathrm{c}$, then for pressure and depth we have the equation

$P = \mathrm{m}d + \mathrm{c}$

where m and c are unknown.

When $d = 0$, $P = 100$, substituting into the equation:

$100 = \mathrm{m} \times 0 + \mathrm{c}$

$\mathrm{c} = 100$

So $P = \mathrm{m}d + 100$

When $d = 10$, $P = 200$, substituting into the equation:

$200 = \mathrm{m} \times 10 + 100$

$\mathrm{m} = 10$

So the equation relating pressure and depth is

$P = 10d + 100$

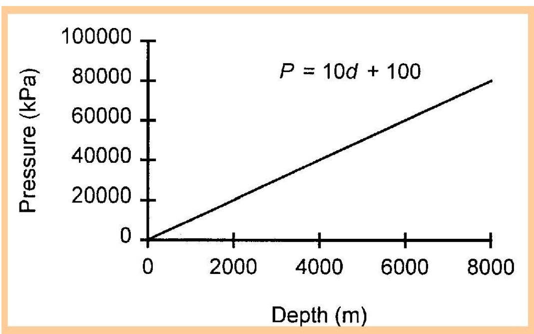
ii. Using this equation $P = 10d + 100$

When $d = 8000$, $P = 10 \times 8000 + 100$

$= 80100$

Pressure at 8000 m is 80100 kPa

iii.



Exercise 3.4

a. The number of chirps made by a cricket changes with temperature and is thought to be best represented by a linear function. If a biologist observed that at 10°C the cricket made 40 chirps a minute while at 15°C they chirped at twice that rate.

   i. Write an expression for the relationship between temperature and chirp rate

   ii. Graph the equation of the function

b. When a mercury thermometer is compared (calibrated) with an electrical resistance thermometer we find that the relationship between the two measurements is a linear function. Temperatures recorded from each thermometer are:

Electrical °C	10	15	20	25	30
Mercury °C	11.20	15.95	20.70	25.45	30.20

   Determine a linear equation that would represent this relationship.

c. The police radar detector reads 60 km h^{-1} when your speedometer reads 55 km h^{-1} and 123 km h^{-1} when your speedometer reads 110 km h^{-1}.

   i. Write a linear equation that would model this relationship.

   ii. What speed should you travel at so that you are not caught for speeding in the 110 km h^{-1} zone?

Solutions

a. i If the relationship is a linear function it should be of the form $y = mx + c$.
Let temperature in °C be T, and rate of chirps in chirps per minute be R.

Then $R = mT + c$

When $T = 10$, $R = 40$ gives $40 = m \times 10 + c$ ←①

When $T = 15$, $R = 80$ gives $80 = m \times 15 + c$ ←②

You need to solve these two equations simultaneously to get a value for m and c (see section 3.3 for method on how to solve simultaneous equations)

To solve for m subtract equation 1 from 2:

$40 = 5m$

$m = 8$

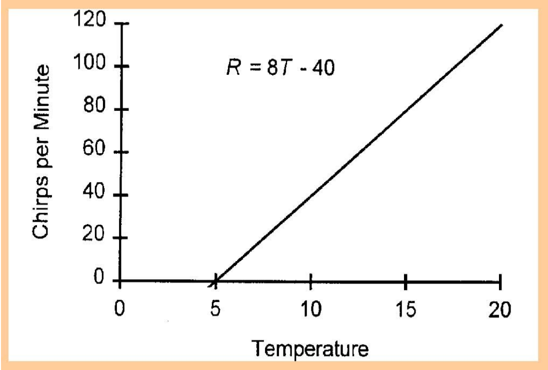
Substitute into 1:

$40 = 10 \times 8 + c$

$c = -40$

Equation is $R = 8T - 40$

ii. Graph of this equation:



Graph of chirps per minute (R) at different temperatures (T °C)

b. If this is a linear function it will be of the form $y = mx + c$. We could substitute the data into this equation and solve for m and c as in the previous question. Alternatively we could use the two point formula to find m (gradient). Let electric temperature be E and mercury temperature be M. Then the linear relationship is

$M = mE + c$

$$\text{Gradient, } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{15.95 - 11.20}{15 - 10}$$
$$= 0.95$$

(Check this for other points).

The equation is then $M = 0.95E + c$.

Substitute in another point $E = 25$ and $M = 25.45$ (say).

$$25.45 = 0.95 \times 25 + c$$

$$c = 1.7$$

The equation is then $M = 0.95E + 1.7$.

(Note a technique called Linear Regression can also be used. It is particularly useful for more variable data).

c. i If the relationship between radar speed and your speed is linear then it will be of the form $y = \mathrm{m}x + \mathrm{c}$. In this case let radar speed be R in km h^{-1} and your speed be S in km h^{-1}. The linear relationship must be

$R = \mathrm{m}S + \mathrm{c}$

If we consider $R = 60$ and $S = 55$, and $R = 123$ and $S = 110$ to be two sets of coordinates (55,60) and (110,123), we can use the two point form of the gradient to find m.

Gradient, $\mathrm{m} = \dfrac{123 - 60}{110 - 55} = 1.15$ (correct to two decimal places)

So $R = 1.15S + \mathrm{c}$

When $S = 55$ and $R = 60$

$60 = 1.15 \times 55 + \mathrm{c}$

$\mathrm{c} = -3.25$

So $R = 1.15S - 3.25$.

ii. When radar speed $R = 110$

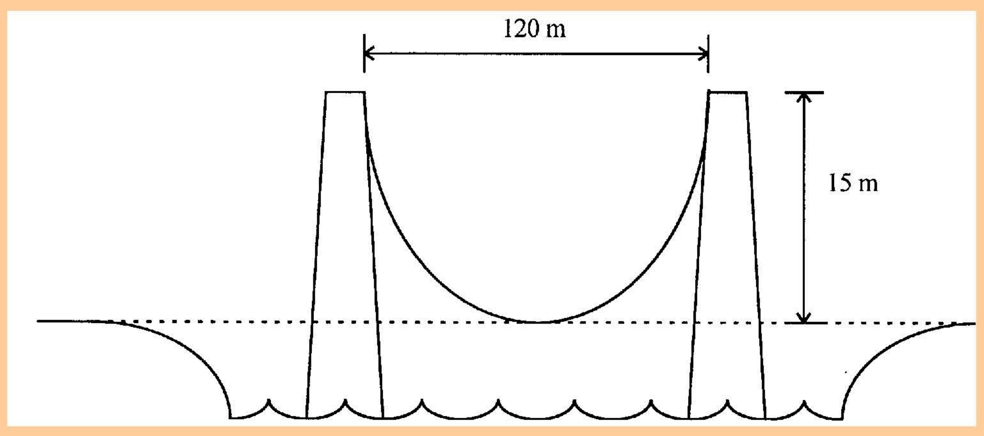
$110 = 1.15S - 3.25$

$S = 98.48 \approx 98$ to the nearest kilometre

Your speed should be 98 km h^{-1}.

3.2.3 The quadratic function

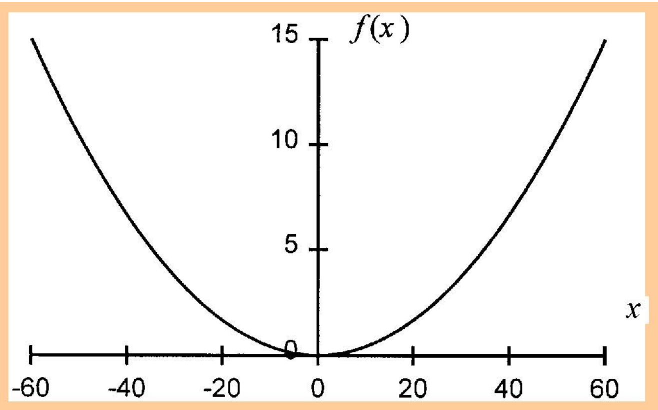
Consider a cable of a suspension bridge. It is suspended between 2 pylons 15 m above the roadway and 120 m apart.



If we measure the distance of the cable from the centre of the road when the cable is a certain height above the road we get

Distance from centre (x m)	–60	–30	–15	0	15	30	60
Height above road ($f(x)$ m)	15	4	1	0	1	4	15

When this is graphed we get



This function can be approximately modelled by a quadratic function

$$f(x) = \frac{3}{722}x^2$$

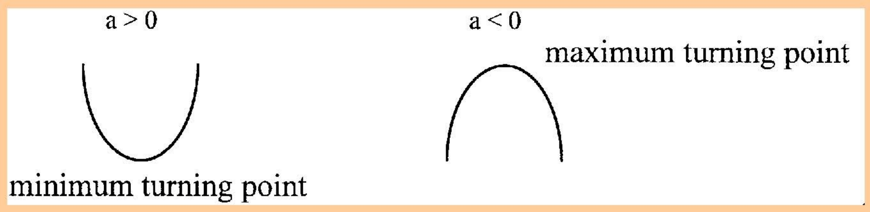
This type of function is called a **parabola**.

The general form of a quadratic or parabolic function is

$$y = ax^2 + bx + c$$

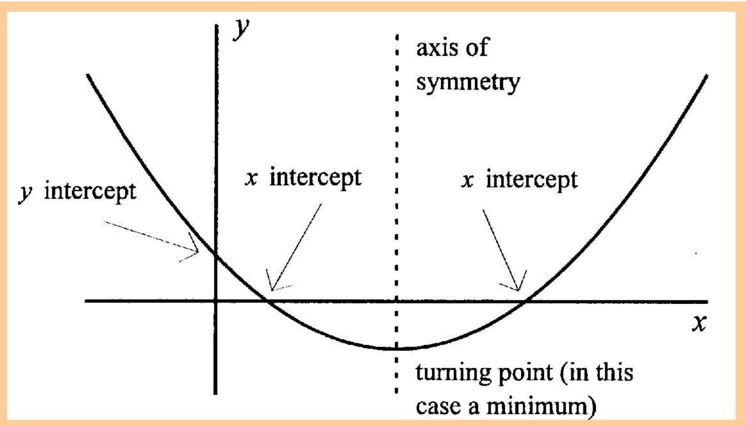
where y is a function of x, and a, b, and c are constants.

The parabola can take two forms.



(Note that when $a = 0$ we have a straight line, not a parabola).

These functions are characterised by their typical shape and the presence of a line (or axis) of symmetry which divides the parabola into two equal halves.



In order to sketch a parabola you need to know

i. where it cuts the y-axis

ii. where it cuts the x-axis (if at all)

iii. the coordinates of the turning point.

Examples

Sketch the following parabolas:

a. $y = 2x^2$

b. $y = x^2 + 5x + 6$

c. $y = 2x^2 - 4x + 5$

d. $y = 1 - 4x - 4x^2$

Solutions

a. $y = 2x^2$

$a > 0$, realise this parabola is ∪-shaped.

To determine where curve cuts y-axis put $x = 0$

When $x = 0$, $y = 2 \times 0^2 = 0$

y-intercept: $y = 0$

To determine where curve cuts x-axis put $y = 0$

When $y = 0$, $y = 2x^2$

$2x^2 = 0$

$x = 0$

x-intercept: $x = 0$

To determine coordinates at turning point:

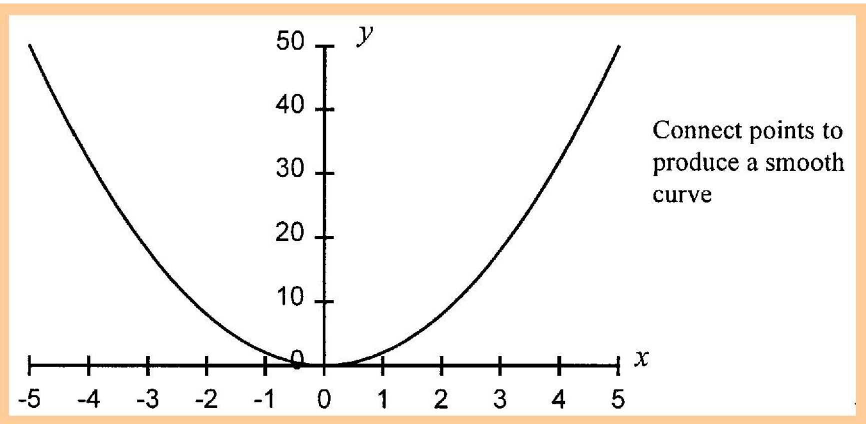
The x-coordinate of the turning point lies on the axis of symmetry. **For the general parabolic equation, $y = ax^2 + bx + c$, the axis of symmetry is given by**

$x = -\dfrac{b}{2a}$ (calculus can also be used to determine this point).

So for turning point, $x = 0$, $y = 0$.

Complete sketch by plotting some extra points at (say) $x = –2$, $x = –1$, $x = 1$ and $x = 2$. (select values around the turning point)

x	–2	–1	0	1	2
y	8	2	0	2	8



b. $y = x^2 + 5x + 6$

$a > 0$, so realise this parabola is ∪-shaped.

y-intercept: $x = 0, y = 6 \Rightarrow (0,6)$

x-intercept: $y = 0, x^2 + 5x + 6 = 0$ See sections 2.2.3 and 2.4.3 on factorising quadratic equations

$$(x + 3)(x + 2) = 0$$

$$x = -3, x = -2$$

$$\Rightarrow (-3,0), (-2,0)$$

Turning point: $x = \dfrac{-b}{2a} = -\dfrac{5}{2\times 1} = -\dfrac{5}{2}$

$$y = \left(-\frac{5}{2}\right)^2 + 5\times\left(-\frac{5}{2}\right) + 6$$

$$= -\frac{1}{4}$$

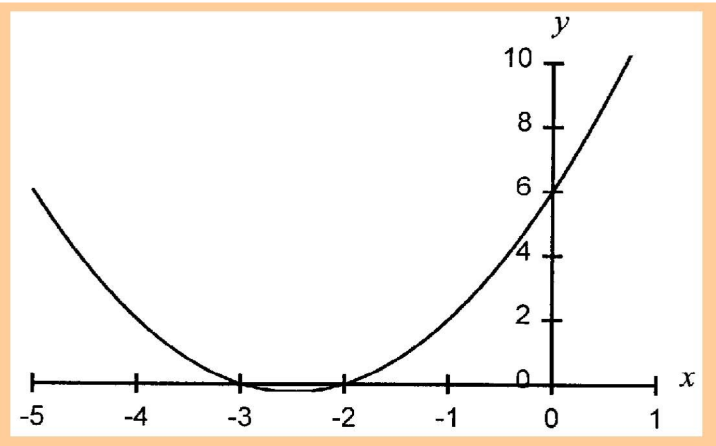
$$\Rightarrow \left(-\frac{5}{2}, -\frac{1}{4}\right)$$

Some extra points:

x	–5	–4	–3	–2	–1	0
y	6	2	0	0	2	6

Choose values around the turning point

Connect points to produce a smooth curve.



c. $y = 2x^2 - 4x + 5$

$a > 0$, so realise this parabola is ∪-shaped.

y-intercept: $x = 0, y = 5 \Rightarrow (0,5)$

x-intercept: $y = 0, 2x^2 - 4x + 5 = 0$

Use quadratic formula to solve (see section 2.4.3)

$$\begin{aligned} b^2 - 4ac &= (-4)^2 - 4 \times 2 \times 5 \\ &= -24 \end{aligned}$$

This parabola does not cut x-axis.

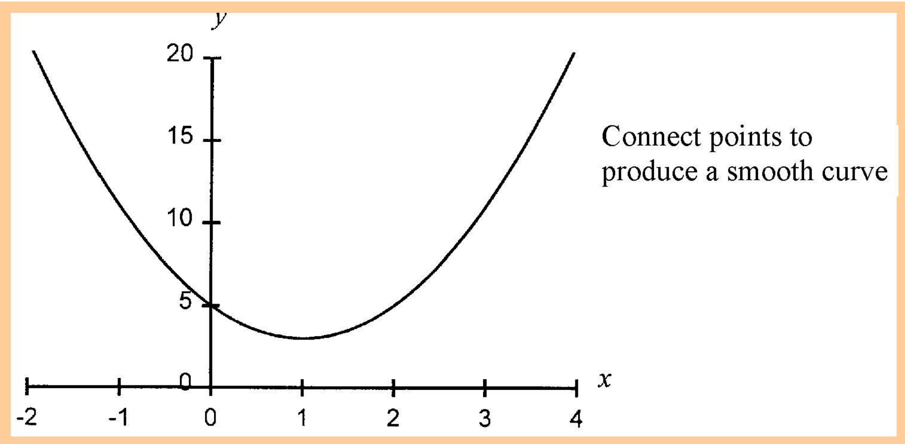
$$\text{Turning point: } x = -\frac{b}{2a} = \frac{4}{2 \times 2} = 1$$

$$\begin{aligned} y &= 2(1)^2 - 4 \times 1 + 5 \\ &= 2 - 4 + 5 \\ &= 3 \\ &\Rightarrow (1, 3) \end{aligned}$$

Some extra points:

x	-2	-1	0	1	2	3
y	21	11	5	3	5	11

Choose values around the turning point



d. $y = 1 - 4x - 4x^2$

$a < 0$, so realise this parabola has ∩-shape.

y intercept: $x = 0, y = 1 \Rightarrow (0,1)$

x intercept: $y = 0,\ 1 - 4x - 4x^2 = 0$

$$-(-1 + 4x + 4x^2) = 0$$
$$4x^2 + 4x - 1 = 0$$
$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-4 \pm \sqrt{4^2 - 4 \times 4 \times (-1)}}{2 \times 4} \\ &= \frac{-4 \pm \sqrt{16 + 16}}{8} \\ &= \frac{-4 \pm 4\sqrt{2}}{8} \\ &= -\frac{1}{2} \pm \frac{1}{2}\sqrt{2} \end{aligned}$$

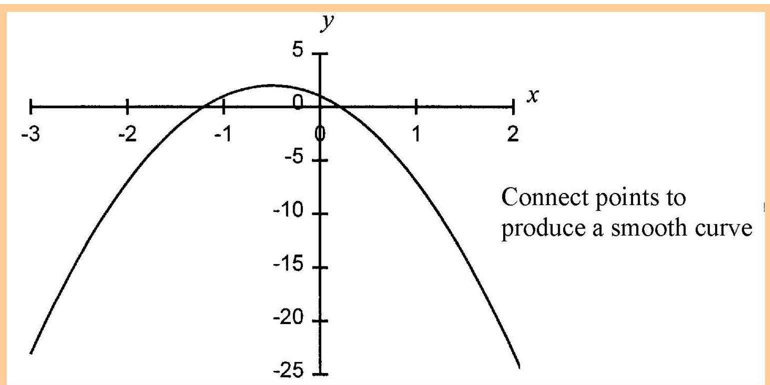
Turning point: $x = -\dfrac{b}{2a} = \dfrac{4}{2 \times -4} = -\dfrac{1}{2}$

$$\begin{aligned} y &= 1 - 4\left(-\frac{1}{2}\right) - 4\left(-\frac{1}{2}\right)^2 \\ &= 2 \\ &\Rightarrow \left(-\frac{1}{2}, 2\right) \end{aligned}$$

Some extra points:

x	–2	–1	0	1	2
y	–7	1	1	–7	–23

Choose values around the turning point



Exercise 3.5

Sketch the following parabolas showing maximum or minimum values.

a. $y = x^2 + 4$

b. $y = 2x^2 - 3$

c. $y = 3x^2 + 2x + 4$

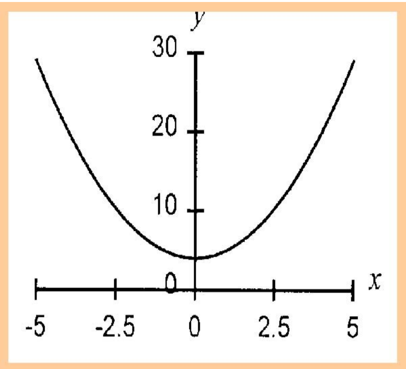
d. $y = -2\,x^2 + 3x + 9$

e. $y = 3(x + 2)^2$

f. $y = 2x^2 - x - 6$

Solutions

a.



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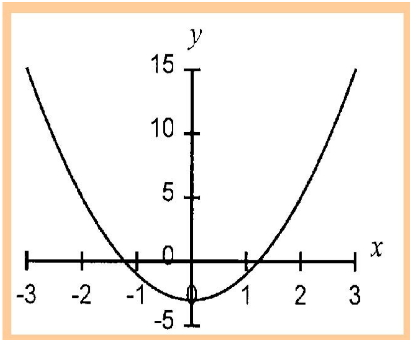

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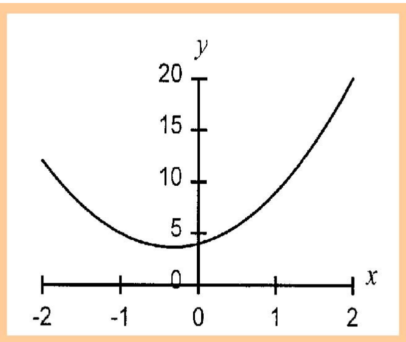

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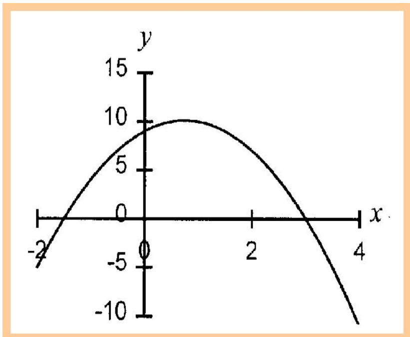

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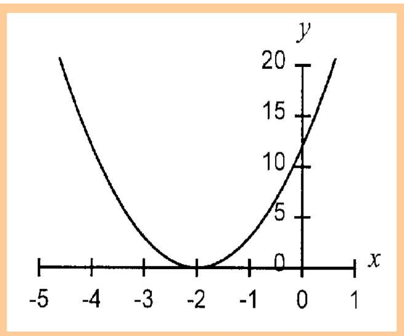

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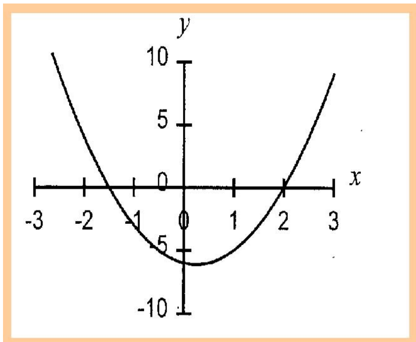


Applications of quadratic functions

Examples

a. The power, P, (in watts) that can be produced in an electrical circuit is a function of the current, I, (in amps) given by the equation $P = -24I^2 + 240I$

   i. Draw a graph of this function

   ii. What is the maximum power that can be produced by this circuit?

b. A rectangle has a perimeter of 24 cm.

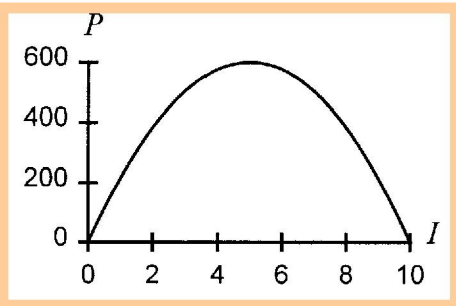
   i. Write an expression, expressing the length of the rectangle in terms of its width.

   ii. Write an equation expressing the area of the rectangle in terms of its width only.

   iii. What dimensions of the rectangle give maximum area?

Solutions

a. i.



where P is the power in watts and I is current in amps

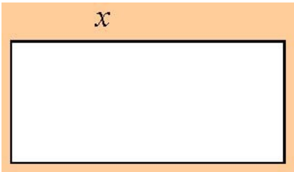
ii. $P = -24I^2 + 240I$

Maximum power will occur at $I = \dfrac{-240}{-2 \times 24} = 5 \quad \left(x = -\dfrac{b}{2a} \right)$

When $I = 5$, $P = -24 \times 5^2 + 240 \times 5 = 600$

Maximum power is 600 watts.

b. i. Let the length of the rectangle be x cm.
If perimeter, $P = 2 \times (\text{length} + \text{width})$



$$24 = 2x + 2W$$
$$2W = 24 - 2x$$
$$W = 12 - x$$

ii. Area, $A = \text{width} \times \text{length}$

$$A = (12 - x) \times x$$
$$A = 12x - x^2$$

iii. Maximum area will occur at the turning point, $x = -\dfrac{b}{2a}$

$$x = \frac{-12}{-2 \times 1} = 6$$

If the length is 6 cm then width is 6 cm (maximum area occurs when rectangle is a square).

Exercise 3.6

a. We have 20 m of fencing material to fence a paddock. If we use an existing fence for one side of the paddock, what is the maximum area of the plot?

b. A rock is thrown vertically upwards with an initial velocity of 15 ms^{-1}. The approximate height it will reach in t seconds is given by the equation

$$h = -5t^2 + 20t$$

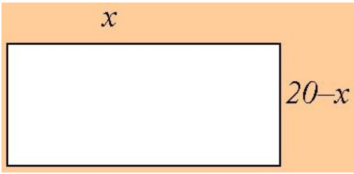
i. Draw a graph of this function.

ii. What is the maximum height reached by the rock?

c. What number when added to its own square will give a minimum sum?

Solutions

a. i. Let the length of the paddock be x, and w be width. Fencing required is



$$20 = x + 2w$$

$$w = \frac{20 - x}{2}$$

Area, A = width × length

$$A = \left(\frac{20 - x}{2}\right) \times x$$

$$A = \left(10 - \frac{1}{2}x\right)x$$

$$A = 10x - \frac{1}{2}x^2$$

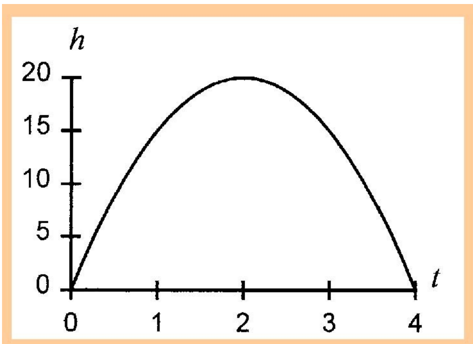
This curve will be parabolic with a maximum occurring at $x = -\dfrac{b}{2a}$

$$x = \frac{-10}{-2 \times \frac{1}{2}} = 10$$

Maximum area $A = 10 \times 10 - \frac{1}{2} \times 10^2 = 50$

Maximum area is 50 square metres when the length is 10 m and the width is 5 m.

b. i. Graph of $h = -5t^2 + 20t$



where h is the height in metres and t is the time in seconds

ii. Maximum height will occur at $x = 2$ (along the axis of symmetry) and will be

$$h = -5 \times 2^2 + 20 \times 2 = 20$$

Maximum height is 20 m.

c. Let the original number be x and the sum be S.

Then $s = x + x^2$

As this function is a parabola with a minimum, this minimum value will occur when

$$x = -\frac{b}{2a} = \frac{-1}{2 \times 1} = -\frac{1}{2}$$

Thus the number to give a minimum sum will be $x = -\frac{1}{2}$.

(The minimum sum is $-\frac{1}{4}$).

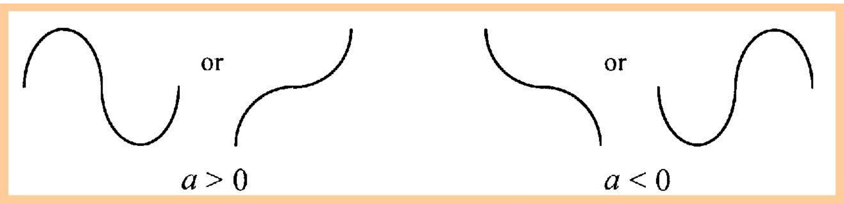
3.2.4 Cubic functions

A cubic function is a function f defined by

$$y = f(x) = ax^3 + bx^2 + cx + d$$

where a, b, c and d are constants and $a \neq 0$.

Cubic functions can take 4 basic forms:



Note if $a = 0$ the function is parabolic, not cubic.

In order to sketch a cubic function it helps to know

i. Where it cuts the y-axis
ii. Where it cuts the x-axis
iii. Coordinates of the turning points (this is best done using calculus and is not a part of this refreshment. See curve sketching in *Engineering mathematics* material).

Examples

Sketch the following curves.

a. $y = x^3 - x$

b. $y = -x^3 + 2x^2 + x - 2$

Solutions

a. $y = x^3 - x$

$a > 0$: Realise that this curve is  or  shaped.

To determine where curve cuts y-axis, put $x = 0$.

When $x = 0$, $y = 0 \Rightarrow (0,0)$

To determine where curve cuts x-axis, put $y = 0$.

When $y = 0$, $x^3 - x = 0$

$$x(x^2 - 1) = 0$$

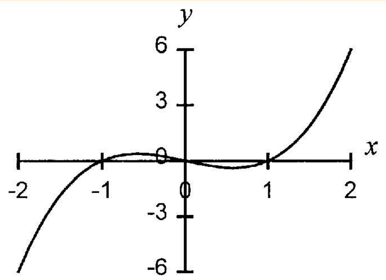
$$x(x + 1)(x - 1) = 0$$

$$x = 0, x = 1, x = -1$$

$\Rightarrow (0,0), (1,0), (-1,0)$

Some extra points:

x	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	-7	$-1\frac{7}{8}$	0	$\frac{3}{8}$	0	$-\frac{3}{8}$	0	$\frac{7}{8}$	6



Note this is just a sketch of the graph, and we can't determine exactly where the turning points occur.

b. $y = -x^3 + 2x^2 + x - 2$

$a < 0$: Realise that this curve is  or  shaped.

To determine where curve cuts y-axis, put $x = 0$.

When $x = 0$, $y = -2 \Rightarrow (0,-2)$

To determine where curve cuts x-axis, put $y = 0$.

$-x^3 + 2x^2 + x - 2 = 0$

$x^3 - 2x^2 - x + 2 = 0$

(Use guess and check and division to solve this equation, see section 2.4.4).

$x = 1$, $1 - 2 - 1 + 2 = 0$ $\quad$ $x = 1$ is a solution

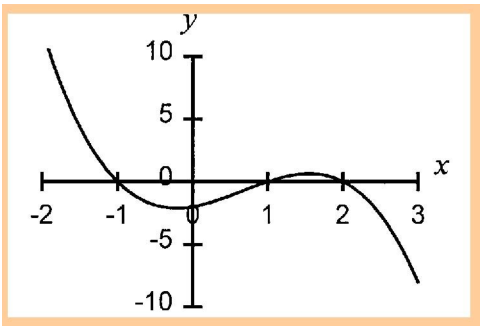
$x = 2$, $8 - 8 - 2 + 2 = 0$ $\quad$ $x = 2$ is a solution

$x = -1$, $-1 - 2 + 1 + 2 = 0$ $\quad$ $x = -1$ is a solution

So points of intersection with the x-axis are (1,0), (2,0) and (–1,0).

Some extra points:

x	–3	–2	–1	0	1	2
y	40	12	0	–2	0	0



Exercise 3.7

Sketch the following curves.

a. $y = x^3$

b. $y = -x^3$

c. $y = x(x - 1)(x + 1)$

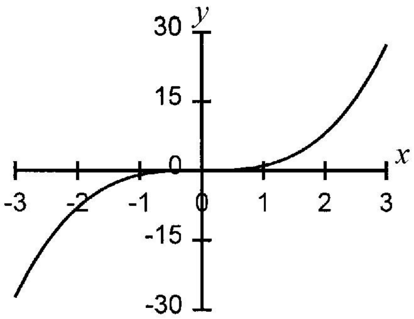
d. $y = -x(x - 1)(x + 1)$

e. $y = x^3 + 2x^2 - 5x - 6$

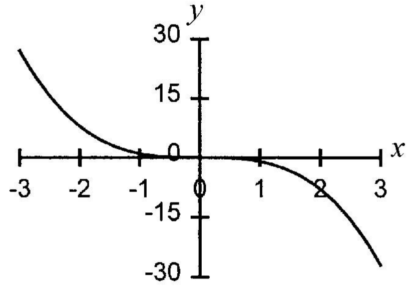
f. $y = x^3 + x^2 - 5x + 3$

Solutions

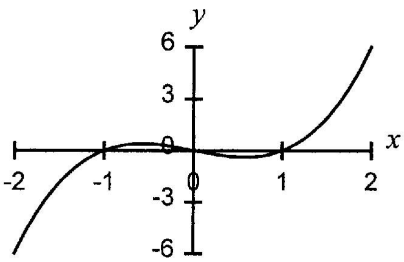
a.



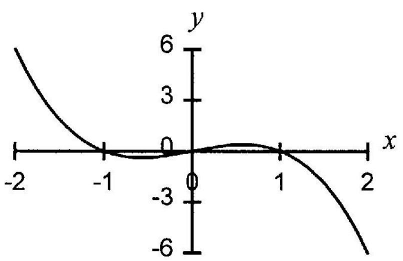
b.



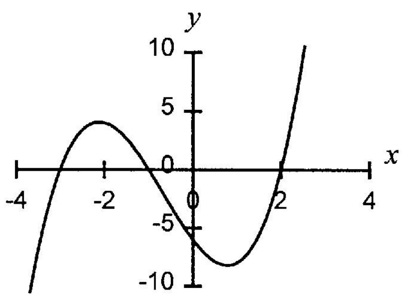
c.



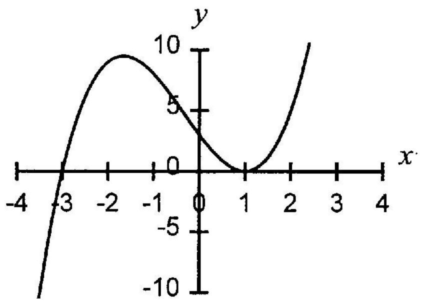
d.



e.



f.



3.2.5 The circle

If we take a piece of string, hold one end and move the other (keeping it tight) in one direction until you reach the starting point, the free end has drawn a circle.

If the string has a length of r the equation of the circle will be

$$x^2 + y^2 = r^2 \qquad \text{where } (0,0) \text{ is the centre and } r \text{ is the radius}$$

In the general case the equation of a circle is

$$(x - a)^2 + (y - b)^2 = r^2 \qquad \text{where } (a,b) \text{ is the centre and } r \text{ is the radius}$$

Notice that a circle is **not** a function, because for some x values there are two values of y. Use the vertical line rule to test this.

To draw a circle you could follow one of two strategies.

a. Plot a number of points.

b. Find the centre and radius and plot some points (usually where it cuts the axes).

Let's investigate the second strategy.

Examples

For the following circles find the centre, radius and x and y intercepts. Use the above to sketch the relations.

a. $(x + 1)^2 + (y - 2)^2 = 5$

b. $x^2 + y^2 - 4x + 2y + 4 = 0$

Solutions

a. For $(x + 1)^2 + (y - 2)^2 = 5$

Because this equation is in the general form, centre is $(-1,2)$ and radius is $\sqrt{5}$ units. (≈ 2.236).

For x-intercept put $y = 0$

$(x + 1)^2 + (0 - 2)^2 = 5$

$x^2 + 2x + 1 + 4 = 5$

$x^2 + 2x = 0$

$x(x + 2) = 0$

$x = 0$, $x = -2$ are the x-intercepts.

For y-intercepts put $x = 0$

$(0 + 1)^2 + (y - 2)^2 = 5$

$1 + y^2 - 4y + 4 = 5$

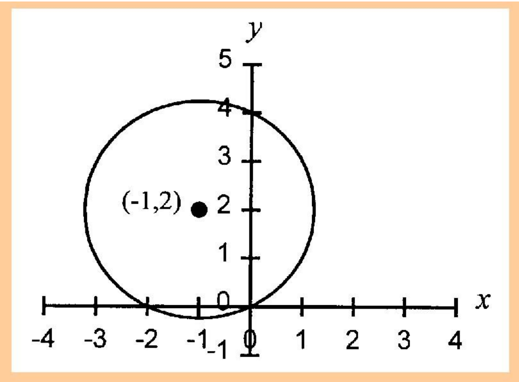
$y^2 - 4y + 5 = 5$

$y^2 - 4y = 0$

$y(y - 4) = 0$

$y = 0$, $y = 4$ are the y-intercepts.

$(x + 1)^2 + (y - 2)^2 = 5$



b. For $x^2 + y^2 - 4x + 2y + 4 = 0$

Equation is not in general form so use technique of ‘completing the square’ to convert to general form.

Completing the square

The technique of completing the square involves converting an expression like $x^2 + 4x$ into $(x + 2)^2 - 4$.

This technique has a wide range of applications including finding the centre of a circle and later in integration.

To do the above we have to use our knowledge of special forms (perfect squares).

$$(x + b)^2 = x^2 + 2bx + b^2$$
$$(x - b)^2 = x^2 - 2bx + b^2$$

Consider $x^2 + 4x$. This is not in the form $x^2 + 2bx + b^2$, although the first two terms are very close to it.

To convert:

$x^2 + 4x$

$= x^2 + 2 \times 2 \times x$	Rewrite expression so that first two terms are same form as $x^2 + 2bx$ and $b = 2$.
$= x^2 + 2 \times 2 \times x + 2^2 - 2^2$	To put in form $x^2 + 2bx + b^2$, add 2^2 then subtract 2^2 so that overall value of expression is not changed.
$= (x^2 + 2 \times 2 \times x + 2^2) - 2^2$	Group first three terms
$= (x + 2)^2 - 4$	Rewrite first three terms in perfect square form where $a = x$ and $b = 2$ (check by expanding $(x + 2)^2$)
So $x^2 + 4x = (x + 2)^2 - 4$	

Let's use this technique to find the centre and radius of

$$x^2 + y^2 - 4x + 2y + 4 = 0$$

Consider

$x^2 + y^2 - 4x + 2y + 4$

$= x^2 - 4x + y^2 + 2y + 4$	group like terms
$= (x^2 - 2 \times 2x) + y^2 + 2y + 4$	rewrite x terms so in same form as $x^2 + 2bx$
$= (x^2 - 2 \times 2x + 2^2 - 2^2) + y^2 + 2y + 4$	add then subtract 2^2 to force expression into $x^2 + 2bx + b^2$ form
$= (x^2 - 2 \times 2x + 2^2) - 2^2 + y^2 + 2y + 4$	group first 3 terms
$= (x - 2)^2 - 2^2 + y^2 + 2y + 4$	rewrite first 3 terms in $(x - b)^2$ form
$= (x - 2)^2 + y^2 + 2y$	add constants
$= (x - 2)^2 + y^2 + 2 \times 1 \times y$	rewrite y terms so in same form as $x^2 + 2bx$
$= (x - 2)^2 + y^2 + 2 \times 1 \times y + 1^2 - 1^2$	add then subtract 1^2
$= (x - 2)^2 + (y^2 + 2 \times 1 \times y + 1^2) - 1^2$	group terms as shown
$= (x - 2)^2 + (y + 1)^2 - 1^2$	rewrite in form $(a + b)^2$

So

$$x^2 + y^2 - 4x + 2y + 4 = 0$$

becomes

$$(x-2)^2 + (y+1)^2 - 1^2 = 0$$

$$(x-2)^2 + (y+1)^2 = 1^2$$

Centre is (2,–1), radius is 1 unit.

To find x-intercept, put $y = 0$

$$(x-2)^2 + 1^2 = 1^2$$

$$(x-2)^2 = 0$$

$$x - 2 = 0$$

$x = 2$ is the x-intercept.

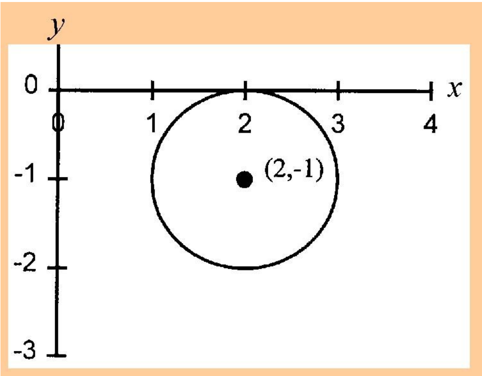
To find y-intercept, put $x = 0$

$$(-2)^2 + (y+1)^2 = 1^2$$

$$4 + y^2 + 2y + 1 = 1^2$$

$$y^2 + 2y + 4 = 0$$

No real solution to this equation because $b^2 - 4ac < 0$, so no y-intercepts.



Exercise 3.8

Find the position of the centre and the radius of the circles:

a. $(x+1)^2+(y-2)^2=5$ b. $x^2+y^2-4x+y+4=0$

Solutions

a. Centre = (–1,2), radius $\sqrt{5} \cong 2.236 =$ units

b. $x^2+y^2-4x+y+4=0$

$x^2-4x+y^2+y+4=0$

It is necessary to 'complete the square' for x^2-4x and for y^2+ y.

$$\left[(x-2)^2-(-2)^2\right]+\left[\left(y+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right]+4=0$$

$$\left[(x-2)^2-4\right]+\left[\left(y+\frac{1}{2}\right)^2-\frac{1}{4}\right]+4=0$$

$$(x-2)^2+\left(y+\frac{1}{2}\right)^2-4-\frac{1}{4}+4=0$$

$$(x-2)^2+\left(y+\frac{1}{2}\right)^2-\frac{1}{4}=0$$

$$(x-2)^2+\left(y+\frac{1}{2}\right)^2=\frac{1}{4}$$

$\therefore$ Centre = $\left(2,-\frac{1}{2}\right)$. Radius = $\sqrt{\frac{1}{4}}=\frac{1}{2}$ units

Examples

Example

Find the points at which the circle $(x-2)^2 + (y-1)^2 = 5$ intersects the axes.

Solution

The circle cuts the x-axis when $y = 0$,
i.e. $(x-2)^2 + (-1)^2 = 5$ i.e. $(x-2)^2 + 1 = 5$

$\therefore (x-2)^2 = 4 \therefore x - 2 = \pm 2$

$\therefore x = 2 + 2 = 4$ or $x = 2 - 2 = 0$

$\therefore$ x-axis is cut at $x = 0$ and $x = 4$

The circle cuts the y-axis when $x = 0$,
i.e. $(-2)^2 + (y-1)^2 = 5$ i.e. $4 + (y-1)^2 = 5$

$\therefore (y-1)^2 = 1 \therefore y - 1 = \pm 1$

$\therefore y = 1 + 1 = 2$ or $y = 1 - 1 = 0$

$\therefore$ y-axis is cut at $y = 0$ and $y = 2$

Example

Find the equation of the circle which passes through the points (0,0) (0,3) (1, 0).

Solution

The general equation to a circle is $(x-a)^2 + (y-b)^2 = r^2$

Given $x = 0, y = 0 \therefore (-a)^2 + (-b)^2 = r^2 \therefore a^2 + b^2 = r^2$

Given $x = 0, y = 3 \therefore (-a)^2 + (3-b)^2 = r^2$

$\therefore a^2 + (3-b)^2 = r^2$ but $r^2 = a^2 + b^2$ from above

$\therefore (3-b)^2 = b^2$

$\therefore 9 - 6b + b^2 = b^2$

$\therefore 9 - 6b = 0$

$\therefore 9 = 6b$

$$\therefore b = \frac{9}{6} = 1.5$$

Given $x = 1$, $y = 0$ $\therefore (1 - a)^2 + (-b)^2 = r^2$ but $r^2 = a^2 + b^2$ as before

$\therefore (1 - a)^2 + b^2 = a^2 + b^2$

$\therefore (1 - a)^2 = a^2$

$\therefore 1 - 2a + a^2 = a^2$

$\therefore 1 - 2a = 0$

$\therefore 1 = 2a$

$\therefore a = 0.5$

$\therefore$ Centre of circle is $(0.5,1.5)$

Now $r^2 = a^2 + b^2$

$\therefore r^2 = 0.5^2 + 1.5^2$

$= 0.25 + 2.25 = 2.5$

$\therefore$ Equation of circle is $(x - 0.5)^2 + (y - 1.5)^2 = 2.5$

Exercise 3.9

For each of the following circles find the (i) position of the centre (ii) length of the radius, and (iii) points, if any, where it cuts the axes.

a. $(x - 3)^2 + (y + 2)^2 = 4$

b. $(x + 1)^2 + (y - 2)^2 = 3$

c. $x^2 - 2x + y^2 - 4y = 3$

d. $x^2 + x + y^2 - 6y + 0.25 = 0$

e. $x^2 - 4x + y^2 - 6y = 12$

f. $x^2 - 2x + y^2 - 16y = 0$

Solutions

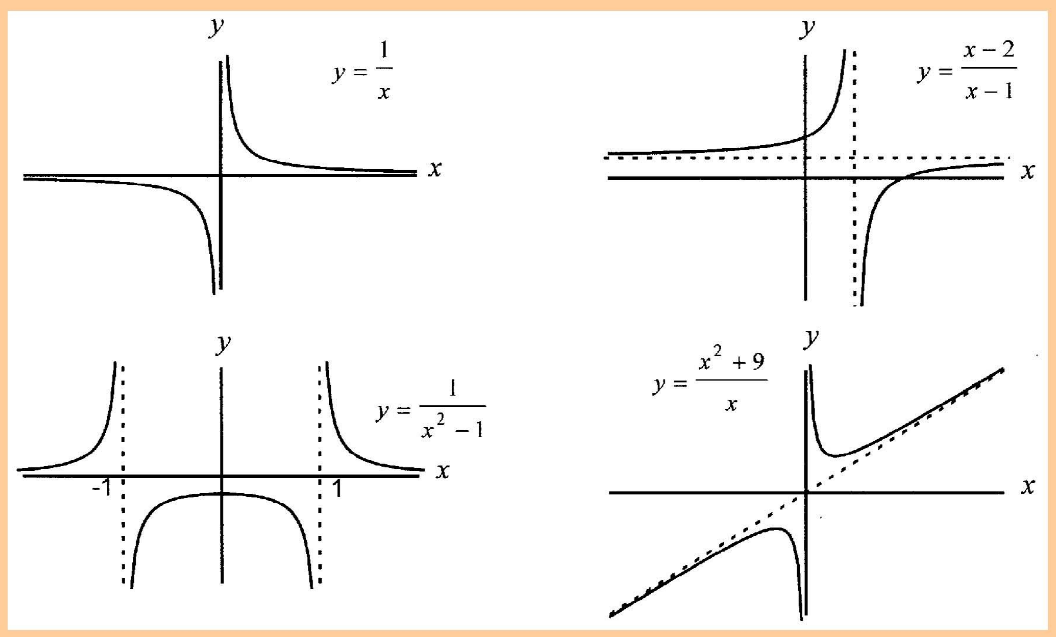
a.	i. $(3,-2)$	ii. $\sqrt{4} = 2$	iii. touches at $x = 3$
b.	i. $(-1,2)$	ii. $\sqrt{3}$	iii. $y = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$
c.	i. $(1,2)$	ii. $\sqrt{8}$	iii. $x = -1$ and $x = 3$, $y = 2 + \sqrt{7}$ and $y = 2 - \sqrt{7}$
d.	i. $(-0.5,3)$	ii. 3	iii. touches at $x = -0.5$ $y = 3 + \sqrt{8.75}$ and $y = 3 - \sqrt{8.75}$
e.	i. $(2,3)$	ii. 5	iii. $x = -2$ and $x = 6$ $y = 3 + \sqrt{21}$ and $y = 3 - \sqrt{21}$
f.	i. $(1,8)$	ii. $\sqrt{65}$	iii. $x = 0$ and $x = 2$, $y = 0$ and $y = 16$.

3.2.6 Rational functions

A rational function $r(x)$ is a function which can be expressed in the form

$$r(x) = \frac{f(x)}{g(x)} \quad \text{where } g(x) \neq 0$$

The graphs of these functions can take a variety of forms.



Characteristically all such graphs possess asymptotes. An asymptote is a straight line which the curve approaches but never actually reaches.

In this course we will investigate sketching only the first two examples of rational functions – rectangular hyperbolas. Later in your study of mathematics you will investigate the other forms of rational functions in more detail.

A hyperbola is an inverse relationship between y and x. It takes the general form:

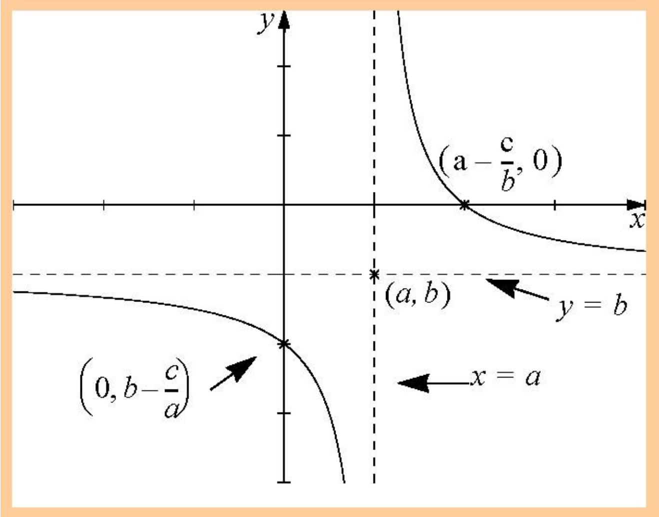
$$y = \frac{c}{x-a} + b$$

When x approaches a, y tends to $+\infty$ or $-\infty$ depending on whether x approaches a from the positive or negative side. Under such conditions we say the curve has a **vertical asymptote** at $x = a$. As y approaches b, x tends to $+\infty$ or $-\infty$ and thus the curve has a horizontal asymptote at $y = b$.

There are two branches to the graph of a hyperbola.

The sketch below shows the important features of a graph of a hyperbola.

[An asymptote is a straight line which the curve approaches but never actually reaches.]



Sketching hyperbolas

To sketch a hyperbola given the general form we need to:

i. determine and plot the asymptotes $x = a$ and $y = b$

ii. determine and plot the points where the curve cuts the y and x-axes, i.e. $\left(0, b - \frac{c}{a}\right)$ and $\left(a - \frac{c}{b}, 0\right)$ respectively

iii. find a few extra points by substituting some x-values and plot these; and

iv. join all points with a smooth curve, bearing in mind that there are two distinct parts of the curve and they are mirror images of each other reflected in a line passing through (a, b) at an angle of either 135° or 45° to the horizontal asymptote.

Exercise 3.10

Sketch the graphs of:

a. $y = \frac{1}{x}$

b. $y = \frac{1}{x - 2} - 5$

c. $y = \frac{3}{1 - x} + 2$

Solutions

a. i. Comparing $y = \frac{1}{x} = \frac{1}{x-0} + 0$ with the general equation of a hyperbola we have $a = 0$, $b = 0$ and $c = 1$.

Thus $x = 0$ is the vertical asymptote and $y = 0$ is the horizontal asymptote.

The point where the asymptotes intersect is $(a,b) = (0,0)$.

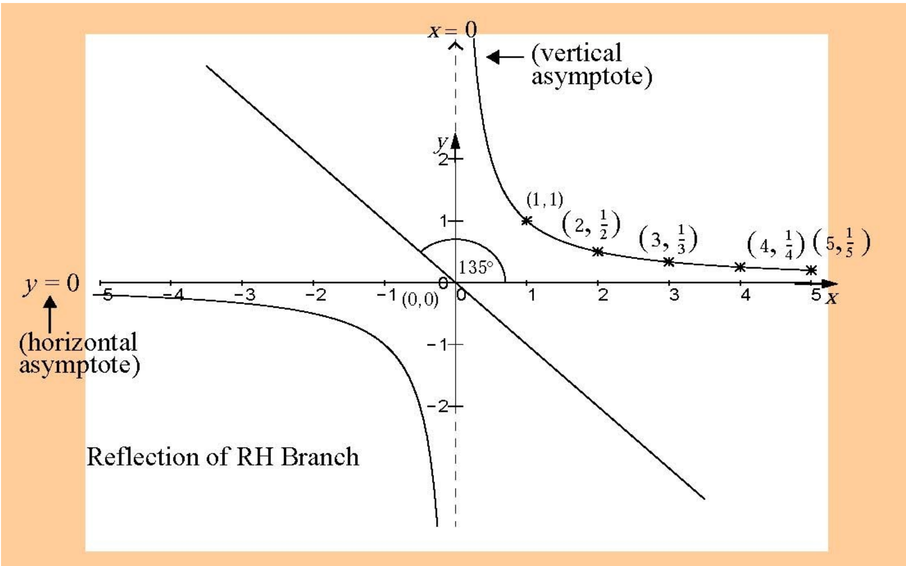
ii. The curve does not cut the x-axis as $a - \frac{c}{b} = 0 - \frac{1}{0}$ which cannot be determined.

The curve does not cut the y-axis as $b - \frac{c}{a} = 0 - \frac{1}{0}$ which also cannot be determined.

(This is obviously the case as the axes are the asymptotes.)

iii.

x	1	2	3	4	5
$y = \frac{1}{x}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$



b. i. Comparing $y = \frac{1}{x-2} - 5$ with general form we have $a = 2$, $b = –5$ and $c = 1$.

Thus $x = 2$ is the vertical asymptote and $y = –5$ is the horizontal asymptote.

The point where the asymptotes intersect is (2,5).

ii The curve cuts the x-axis at $x = a - \dfrac{c}{b}$

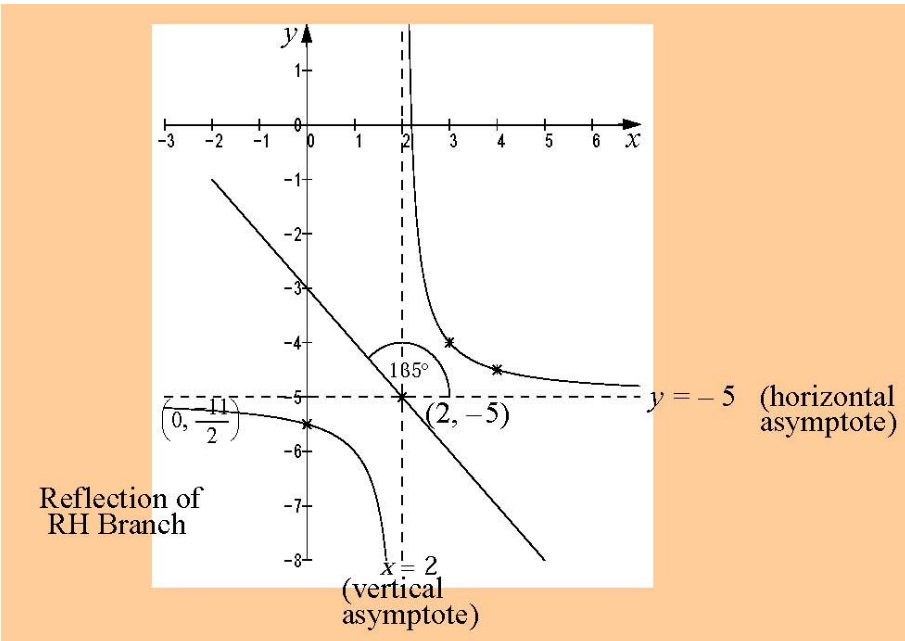
i.e. at $x = 2 - \dfrac{1}{-5} = 2\dfrac{1}{5} \Rightarrow \left(\dfrac{11}{5}, 0\right)$ is on the graph.

The curve cuts the y-axis at $y = b - \dfrac{c}{a}$

i.e. at $y = -5 - \dfrac{1}{2} = -5\dfrac{1}{2} \Rightarrow \left(0, -\dfrac{11}{2}\right)$ is on the graph.

iii.

x	3	4
$y = \dfrac{1}{x-2} - 5$	–4	$-\dfrac{9}{2}$



c. i. Comparing $y = \frac{3}{1-x} + 2 = \frac{-3}{x-1} + 2$ with general form we have $a = 1$, $b = 2$ and $c = -3$.

Thus $x = 1$ is the vertical asymptote and $y = 2$ is the horizontal asymptote.

Point of intersection of asymptotes is (1,2).

ii. When $x = 0$, $y = b - \frac{c}{a} = 2 - \frac{-3}{1} = 5 \Rightarrow (0, 5)$ is on graph

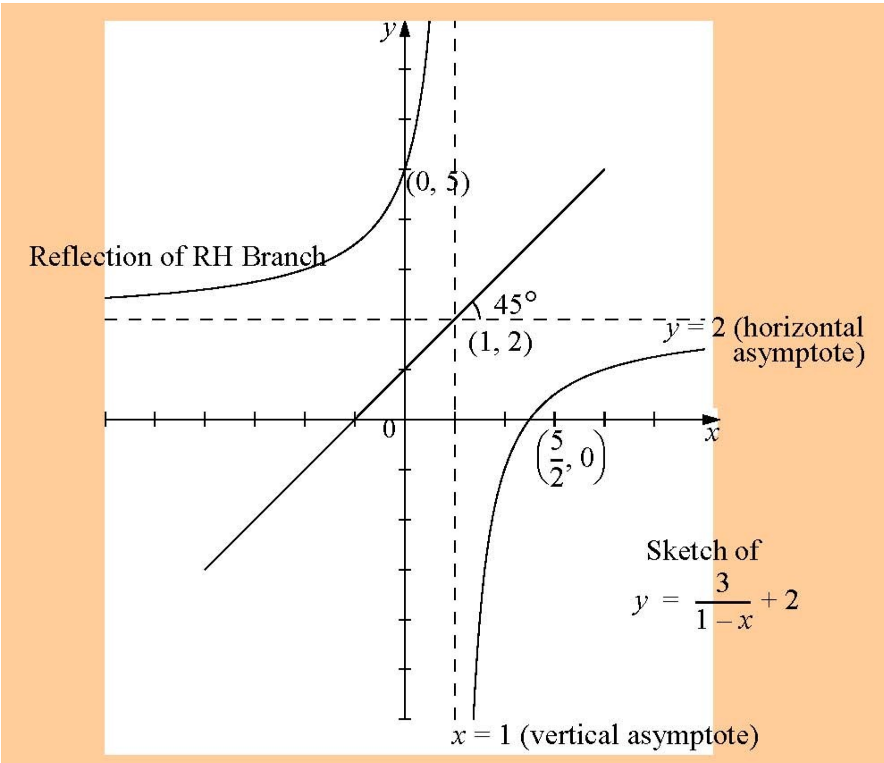
(or put $x = 0$ and solve $y = \frac{3}{1-0} + 2$)

When $y = 0$, $x = a - \frac{c}{b} = 1 - \frac{-3}{2} = 2\frac{1}{2} \Rightarrow \left(\frac{5}{2}, 0\right)$ is on graph

(or put $y = 0$ and solve $0 = \frac{3}{1-x} + 2$)

iii.

x	2	4
$y = \frac{3}{1-x} + 2$	–1	1



Exercise 3.11

Sketch the following hyperbolas indicating vertical and horizontal asymptotes.

a. $y = \dfrac{1}{x-2} + 3$

b. $y = \dfrac{1}{x+3} - 2$

c. $y = \dfrac{2}{x-1} + 3$

d. $y(x-2) = 4$

e. $\dfrac{1}{y-2} = x - 3$

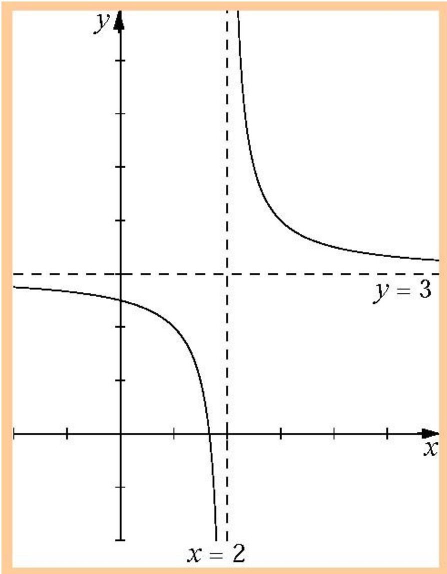
f. $y = 1 - \dfrac{1}{x+3}$

Solutions

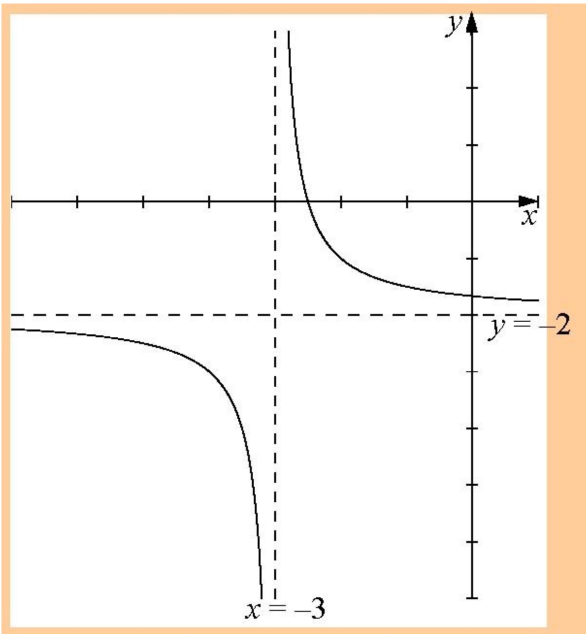
a. Vertical asymptote at $x = 2$

Horizontal asymptote at $y = 3$

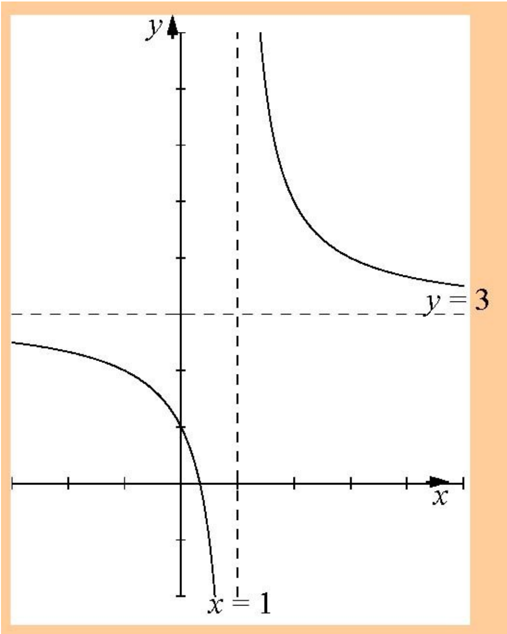
When $x = 3$, $y = 4$



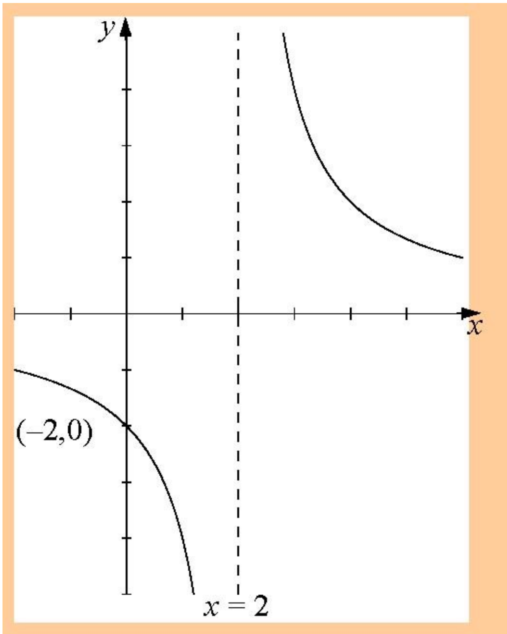
b.



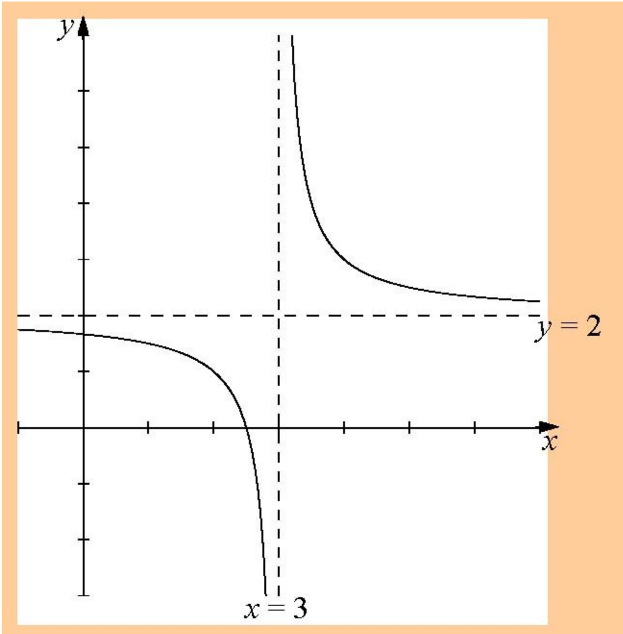
c.



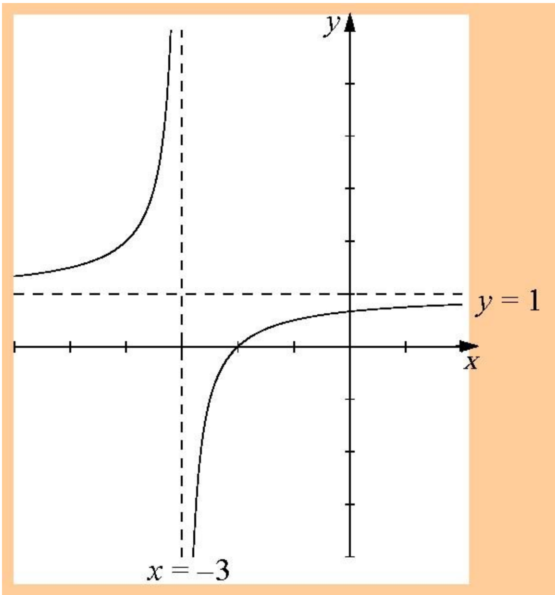
d.



e.



f.



3.2.7 Other functions

Knowing the basic characteristics of particular functions or relations and how to graph in general, we can graph many different functions which are combinations of other functions.

In general, if X and Y are two sets of numbers, a function from X into Y is a correspondence that associates with each element of X a **unique** element of Y. The set X is called the **domain** of the function. For each element x in X, the corresponding y in Y is called the **value of the function** at x (or the image of x). The set of all functional values (i.e. y values) is called the **range** of the function. We have been representing functions by ordered pairs in module 3. e.g. if $y = 3x + 4$ we say that y is a function of x. The ordered pairs, (0,4), (1,7) and all other points that lie on the line $y = 3x + 4$ are valid realisations of the function. We could write this statement as F = $\{(x,y) \mid y = 3x + 4\}$ which we read as F is the function that provides the ordered pairs (x,y) such that $y = 3x + 4$. In this example the domain of the function (i.e. possible x values) is $(-\infty, \infty)$ and the corresponding y values (or range of the function) is also $(-\infty, \infty)$.

Consider the function given by $\{(x,y) \mid y = \sqrt{x+1}\}$. The square root sign puts a restriction on the values of x that can be used, and, consequently, on the values of y. The number under the square root sign, must never be negative, as the real number system does not allow square roots of negative numbers. (There is no real number whose square is a negative number.)

Hence, when $y = \sqrt{x+1}$, $x + 1$ must be zero or positive.

i.e. $x + 1 \geq 0 \quad \therefore x \geq -1$.

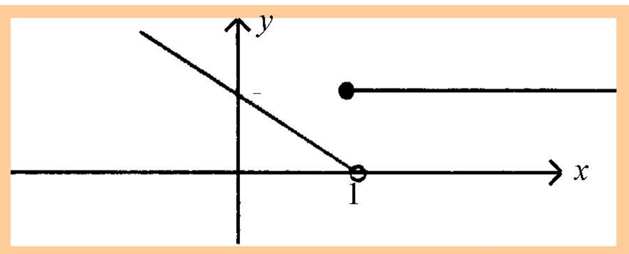
Thus, the domain of the function is $[-1,\infty)$. The smallest functional value is $y = 0$ and the range of the function is $[0,\infty)$.

Do not confuse the notation for the intervals of the range and the domain with ordered pairs.

A function $y = f(x)$ that can be graphed over its domain with one continuous motion of the pen is an example of a continuous function.

The parabola $y = 2x^2$ is a continuous function.

However $f(x) = \begin{cases} 1 - x, x < 1 \\ 1, x \geq 1 \end{cases}$ is discontinuous at the point $x = 1$, because at this point there is a break in the graph.



Sketch of $f(x) = \begin{cases} 1 - x, x < 1 \\ 1, x \geq 1 \end{cases}$

When drawing a graph of a mathematical equation or function, it is helpful if you have some idea of its general shape e.g. circle, parabola, etc. However, graphs of completely strange functions may be drawn by plotting a number of points and connecting them. To do this, select a number of suitable x-values and evaluate the function at each value. This produces a set of corresponding y-values. These points are then plotted and connected. The more points plotted, the greater the accuracy of the curve. Be sure to plot more points in regions of unusual behaviour. If the domain of x-values is not specified, ensure you examine the function as $x \to +\infty$ and $x \to -\infty$.

Note graphs of exponential and logarithmic functions are detailed in module 4.0.

Example

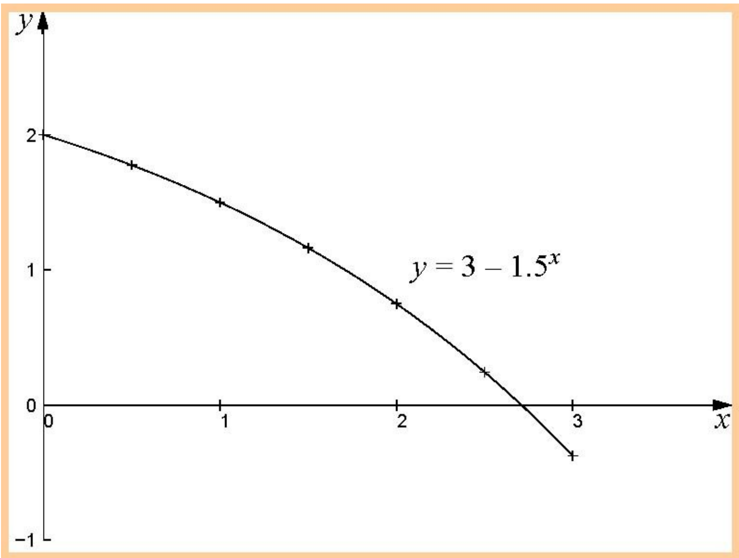
Example

Draw a graph of the function

$$y = 3 - 1.5^x$$

over the domain $x = 0$ to $x = 3$

Solution



x	0	0.5	1	1.5	2	2.5	3
y	2	1.775	1.5	1.163	0.75	0.244	–0.375

Note: The y-values are obtained by substituting the x-values into the function $y = 3 - 1.5^x$. For example, when $x = 0.5$, $y = 3 - 1.5^{0.5} = 3 - \sqrt{1.5} = 3 - 1.225 = 1.775$.

Example

Draw a graph of $y = \dfrac{2}{x-1} + \dfrac{3}{x-2}$

Solution

x	0	0.5	1	1.2	1.5	1.8	2	2.5	3	3.5	4
y	–3.5	–6	$\pm\infty$	6.25	–2	–12.5	$\pm\infty$	7.33	4	2.8	2.17

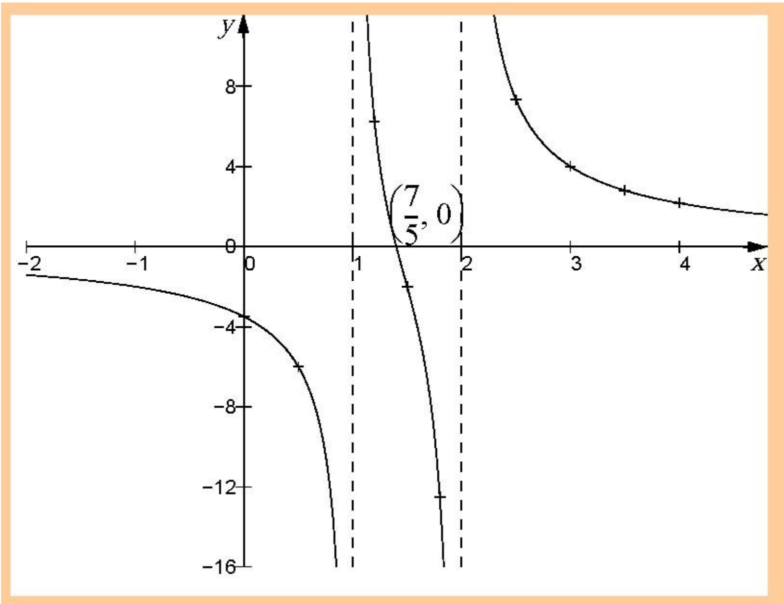
Note that extra points are required in the region $x = 1$ to $x = 2$ since the curve's behaviour is unusual here.

Other points of interest are:

i. when $y = 0$, $\dfrac{2}{x-1} + \dfrac{3}{x-2} = 0$. Solving gives $x = \dfrac{7}{5}$

ii. as $x \to +\infty$, $y \to 0$ from above.

iii. as $x \to -\infty$, $y \to 0$ from below.

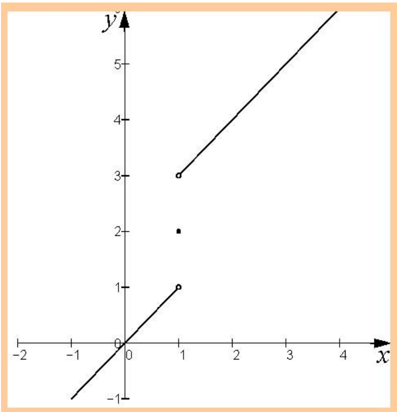


Example

Graph the function $f(x) = \begin{cases} x & \text{if } -1 \le x < 1 \\ 2 & \text{if } x = 1 \\ x + 2 & \text{if } x \ge 1 \end{cases}$

Solution

The domain of f is all the real numbers $x \geq -1$. We use a filled circle ● to indicate that at $x = 1$, the value of f is $f(1) = 2$. We use an open circle ○ to illustrate that the function does not assume 3 or 1 at $x = 1$.



Exercise 3.12

Draw graphs of the following functions over the domains given.

a. $y = \left(\frac{1}{2}\right)^x$, $x = 0$ to 4

b. $y = 2^{2x}$, $x = -1$ to 2

c. $y = 0.5 - 0.3^x$, $x = 0$ to 4

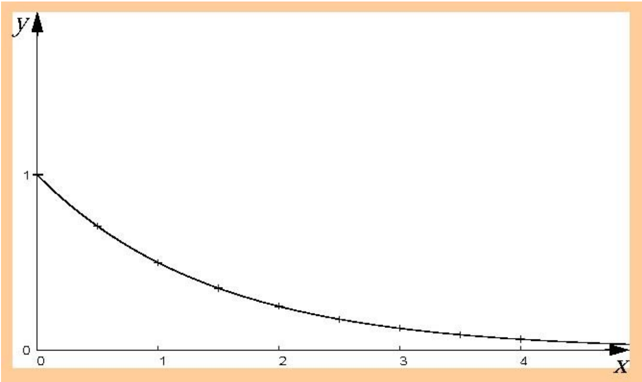
d. $y = \frac{1}{x - 2} + \sqrt{x}$, $x = 0$ to 4

e. $y = 3x^{-1.2} + 1$, $x = 1$ to 3

Solutions

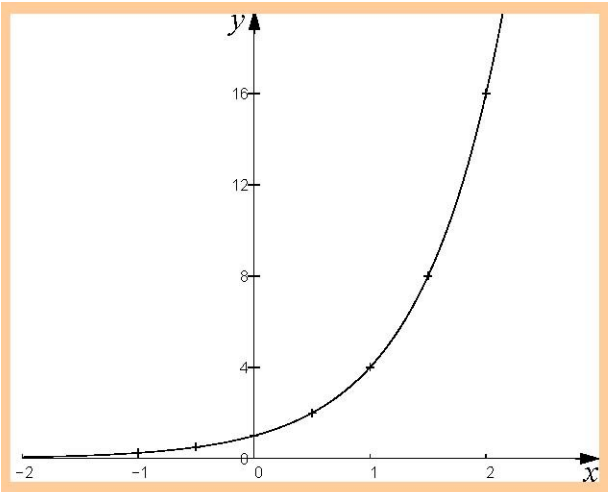
a.

x	0	0.5	1	1.5	2	2.5	3	3.5	4
y	1	0.707	0.5	0.354	0.25	0.177	0.125	0.089	0.062



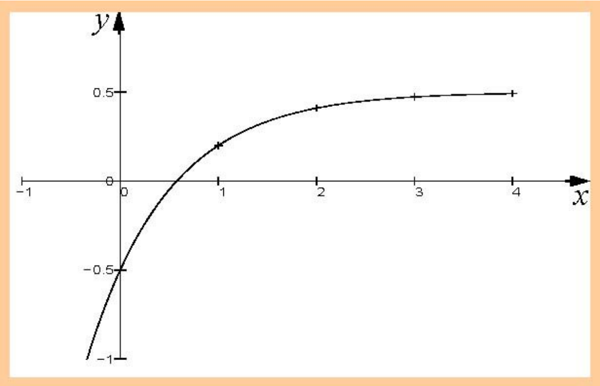
b.

x	–1	–0.5	0	0.5	1	1.5	2
y	0.25	0.5	1	2	4	8	16



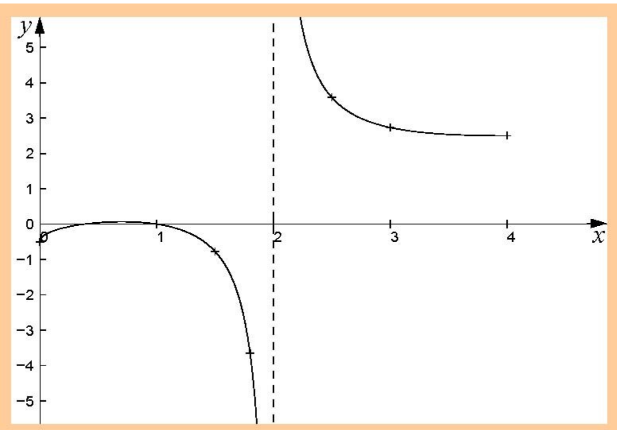
c.

x	0	1	2	3	4
y	–0.5	0.2	0.41	0.473	0.4919



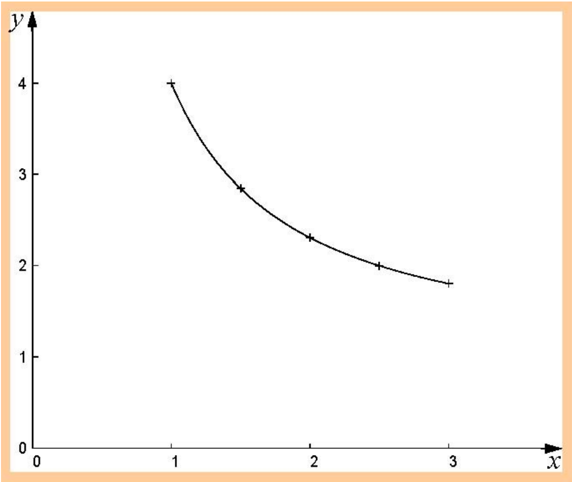
d.

x	0	1	1.5	1.8	2.2	2.5	3	4
y	$-\frac{1}{2}$	0	–0.775	–3.658	6.483	3.581	2.732	2.5



e.

x	1	1.5	2	2.5	3
y	4	2.844	2.306	1.999	1.803



3.3 Intersections of functions

When two or more functions intersect it is important that we are able to find the point or points of intersection. We can attempt to do two things. Find:

1. an exact solution using algebraic methods
2. an approximate solution using graphical methods.

Note the second alternative is only used when an algebraic solution is not possible or when an alternative check is required for the algebraic solution.

3.3.1 Intersection of linear functions

Algebraic solution of equations

This topic is also referred to as solution of simultaneous equations.

i. Two equations in two unknowns

If a problem involves two unknown variables x and y, then a solution will exist only if two separate equations exist involving x and y. A pair of values for x and y which satisfies both equations at once is called a simultaneous solution of the equations. For example the simultaneous solution of $x + y = 7$ and $x - y = 3$ is $x = 5$, $y = 2$.

There are two algebraic methods for solving a set of two equations involving two variables.

Substitution method

Using one of the equations, we express one variable in terms of the other. This expression is then substituted into the other equation to form an equation in one variable only. Find the value of this variable and hence find the value of the other variable. For example, consider

$$2x + y = 21 \qquad (1)$$

$$3x + 4y = 44 \qquad (2)$$

From (1), $y = 21 - 2x$

Substituting in (2), $3x + 4(21 - 2x) = 44$

Solving for x, $3x + 84 - 8x = 44$

$$-5x + 84 = 44$$

$$-5x = -40$$

$$x = 8$$

Solving for y, $\therefore y = 21 - 2x = 21 - 16 = 5$

i.e. The solution is $x = 8$, $y = 5$.

You should check that this solution satisfies **both** the original equations.

Elimination method

If necessary, multiply the equations by such numbers so as to make the x-coefficients equal.

If the x-coefficients have the same sign, subtract the equations. If the signs are opposite, add the equations. Solve the resulting equation involving y only. Substitute the y-value into the original equation to find x. For example, consider

$$2x + y = 21 \qquad (1)$$

$$3x + 4y = 44 \qquad (2)$$

$(1) \times 3 \rightarrow$ $$6x + 3y = 63 \qquad (3)$$

$(2) \times 2 \rightarrow$ $$6x + 8y = 88 \qquad (4)$$

$(3) - (4) \rightarrow$ $$-5y = 63 - 88 = -25$$

$$\therefore y = 5$$

Sub in (1), $$2x + 5 = 21 \therefore 2x = 16 \therefore x = 8$$

i.e. The solution is $x = 8$, $y = 5$.

You should check that this solution satisfies **both** the original equations.

You could follow this same method and make the y-coefficients equal.

Examples

Solve the following sets of equations using both methods

a. $2x - y = 4$ (1)
   $x + y = 5$ (2)

b. $5x + 2y = 3$ (1)
   $2x + 3y = -1$ (2)

Solutions

a. **Substitution method**

From (2) $$y = 5 - x$$

Sub in (1) $$\therefore 2x - (5 - x) = 4$$

$$2x - 5 + x = 4$$

$$3x - 5 = 4 \therefore 3x = 9 \therefore x = 3$$

$$\therefore y = 5 - 3 = 2$$

The solution is $x = 3$, $y = 2$ which satisfies both the original equations.

a. **Elimination method**

	$2x - y = 4$	(1)
(2) × 2 →	$2x + 2y = 10$	(3)
(1) – (3) →	$\therefore -3y = -6 \therefore y = 2$	
Sub in (1)	$\therefore 2x - 2 = 4 \therefore 2x = 6 \therefore x = 3$	

The solution is $x = 3$, $y = 2$.

b. **Substitution method**

From (1) $\quad 2y = 3 - 5x \quad \therefore\ y = \dfrac{(3-5x)}{2}$

Sub in (2) $\quad 2x + \dfrac{3(3-5x)}{2} = -1 \quad \therefore\ 2x + \dfrac{9-15x}{2} = -1$

$\therefore 4x + 9 - 15x = -2 \quad \therefore -11x = -9 - 2 = -11 \therefore x = 1$

$\therefore 2y = 3 - 5 = -2 \therefore y = -1$

The solution is $x = 1$, $y = -1$ which satisfies both the original equations.

b. **Elimination method**

(1) × 2 →	$10x + 4y = 6$	(3)
(2) × 5 →	$10x + 15y = -5$	(4)
(3) – (4) →	$-11y = 6 + 5 = 11 \therefore y = -1$	
Sub in (1)	$\therefore 5x - 2 = 3 \therefore 5x = 5 \therefore x = 1$	

The solution is $x = 1$, $y = -1$.

ii. Three equations in three unknowns

For a set of 3 simultaneous equations involving 3 variables, the elimination method is the only practical procedure. Consider the set:

$2x - y + z = 3$	(1)
$x + 3y - 2z = 11$	(2)
$3x - 2y + 4z = 1$	(3)

Use (1) to eliminate x from (2) and (3)

$(2) \times 2 \rightarrow$	$2x + 6y - 4z$	$= 22$	(4)
$(4) - (1) \rightarrow$	$7y - 5z$	$= 19$	(5)
$(3) \times 2 - (1) \times 3 \rightarrow$	$-y + 5z$	$= -7$	(6)
$(6) \times 7 + (5) \rightarrow$	$30z$	$= -30 \therefore z = -1$	
Sub in (5)	$\therefore 7y + 5$	$= 19 \therefore 7y = 14 \therefore y = 2$	
Sub in (1)	$\therefore 2x - 2 - 1$	$= 3 \therefore 2x = 6 \therefore x = 3$	

i.e. The solution is $x = 3$, $y = 2$, $z = -1$.

Check that the solution satisfies **each** of the original equations.

Examples

Example

Solve the equations

a.	$x - y + 2z = 0$	(1)
	$x - 2y + 3z = -1$	(2)
	$2x - 2y + z = -3$	(3)
b.	$x + y + 2z = 3$	(1)
	$3x - y + z = 1$	(2)
	$2x + 3y - 4z = 8$	(3)

Solutions

a.

(2) – (1) →	$-y+z$	$=-1$	(4)
(3) – 2 × (1) →	$-3z$	$=-3 \therefore z=1$	
Sub in (4)	$-y+1$	$=-1 \therefore y=2$	
Sub in (1)	$x-2+2$	$=0 \therefore x=0$	

i.e. The solution is $x = 0$, $y = 2$, $z = 1$.

Check that this solution satisfies **each** of the original equations.

b.

(2) – 3 × (1) →	$-4y-5z=-8$	(4)
(3) –2 × (1) →	$y-8z=2$	(5)
4 × (5) + (4) →	$-37z=0$	
	$\therefore z=0$	
Sub in (5)	$y=2$	
Sub in (1)	$x+2+0=3$	
	$\therefore x=1$	

i.e. The solution is $x = 1$, $y = 2$, $z = 0$.

Check that the solution satisfies **each** of the original equations.

Example

Bottle A holds a 25% alcohol solution and bottle B has a 40% alcohol solution. How much should we take from each bottle to make 40 mL of 30% solution?

Solution

Let x and y be the number of mL from Bottle A and Bottle B respectively.

$$x + y = 40 \qquad (1)$$

Amount of alcohol is $0.25x + 0.4y = 40 \times 0.3$

$$\therefore 25x + 40y = 1200 \qquad (2)$$

We now have two simultaneous equations in x and y to solve.

$(2) - 25 \times (1) \rightarrow \qquad 40y - 25y = 1200 - 25 \times 40$

$$\therefore 15y = 200$$

$$\therefore y = \frac{40}{3}$$

Sub in (1) $\qquad x + \frac{40}{3} = 40$

$$\therefore x = 40 - \frac{40}{3} = \frac{80}{3}$$

i.e. We need $26\frac{2}{3}$ mL from Bottle A and $13\frac{1}{3}$ mL from Bottle B to make 40 mL of 30% alcohol solution.

Exercise 3.13

Solve the following sets of simultaneous equations.

a. $2x + y = 5$
$x + 2y = 4$

b. $3x - y = 12$
$x + y = 8$

c. $3x - 4y = 5$
$5x - 12y = 3$

d. $x - y + z = 2$
$2x + y - 3z = 5$
$x + 2y - 2z = 5$

e. $3x + 4y - 6z = -13$
$x - y + z = 0$
$2x - y + 3z = 3$

f. $x - 8y + 2z = -1$
$x - 3y + z = 1$
$2x - 11y + 3z = 2$

Solutions

a. $x = 2, y = 1$

b. $x = 5, y = 3$

c. $x = 3, y = 1$

d. $x = 3, y = 2, z = 1$

e. $x = -1.25, y = 0.875, z = 2.125$

f. No solution exists

Graphical solution of equations

Recall that if two lines are parallel to each other, then there is no (x,y) value which will satisfy the equations of both lines, i.e. the equations are **inconsistent**.

Furthermore if the two lines are essentially the same line, i.e. coincident, then there is an infinite number of solutions to the equations of the lines.

We can obtain approximate solutions to linear equations which are not parallel or coincident using the graphical method briefly given below.

Examples

Solve (if possible) the following systems of equations graphically:

a. $2x + y = 5$
$y = 7 + 2x$

b. $x + y = 5$
$x = 2 - y$

Solutions

a. $2x + y = 5 \qquad y = 7 + 2x$

Rearranging yields:

$y = -2x + 5, \qquad y = 2x + 7$

Approximate solution from graph is:

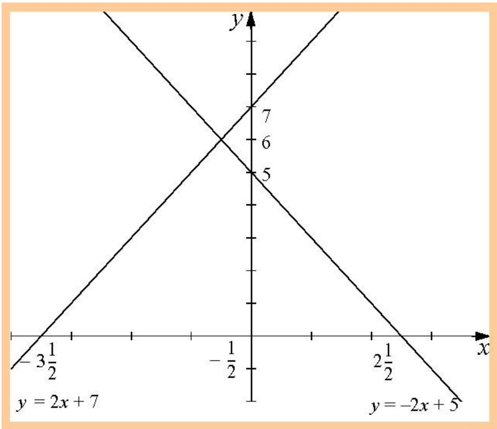
$x \approx -0.5$

$y \approx 6$

Check solution:

$$2\left(-\frac{1}{2}\right) + 6 = 5$$

$$7 + 2\left(-\frac{1}{2}\right) = 6$$



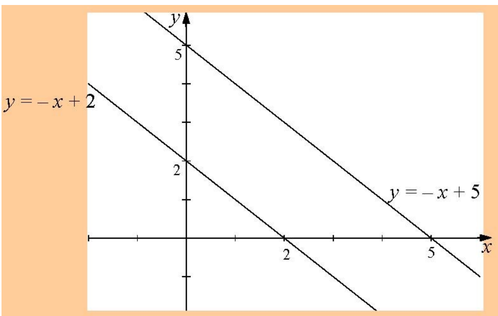
b. $x + y = 5$

$x = 2 - y$

Rearranging yields:

$y = -x + 5, \qquad y = -x + 2$

$\therefore$ Because the two lines are parallel to each other, the equations are inconsistent and there is no solution to the system.



3.3.2 Intersection of functions other than straight lines or planes

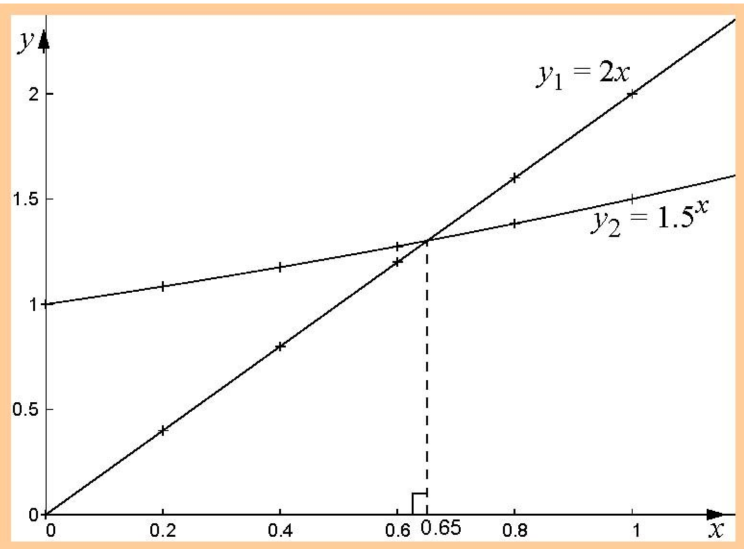
Obviously we would not use the graphical method to obtain approximate solutions to a system of equations if analytical methods which will yield the exact solution are available. However in many situations, we are unable to solve an equation or a system of equations by algebraic means and we have to find an approximate solution by other methods. Often it is possible to get a rough approximation from carefully drawn graphs.

Consider the equation $2x - 1.5^x = 0$. There is no algebraic method for solving this equation.

However, the equation can be rearranged as $2x = 1.5^x$. We can plot two simple functions $y_1 = 2x$ and $y_2 = 1.5^x$ on the same graph. At the point where these graphs intersect, the y-values must be the same i.e. $y_1 = y_2$. Thus, at this point, we have a solution of $2x = 1.5^x$. i.e. we have solved the equation $2x - 1.5^x = 0$. The accuracy of this solution depends on the accuracy with which the graphs were plotted.

These curves obviously intersect between $x = 0$ and $x = 1$, so we select values of x from 0 to 1 and determine the corresponding values of y_1 and y_2.

x	0	0.2	0.4	0.6	0.8	1
$y_1 = 2x$	0	0.4	0.8	1.2	1.6	2
$y_2 = 1.5^x$	1	1.084	1.176	1.275	1.383	1.5



Approximate solution is x ≈ 0.65

Check that this solution satisfies (or nearly satisfies) the original equation.

Examples

Find an approximate solution of the following equations:

a. $x^2 - 3 - \sqrt{x} = 0$

b. $\dfrac{1}{x-1} + \sqrt{2x^2+1} = 5$

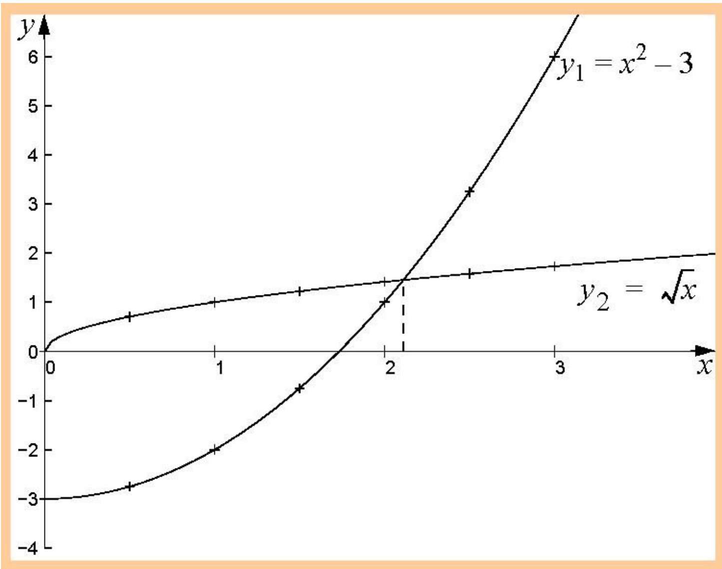
Solutions

a. $x^2 - 3 = \sqrt{x}$

$y_1 = x^2 - 3, \qquad y_2 = \sqrt{x}$

Recognise that $y_1 = x^2 - 3$ is a parabola.

x	0	0.5	1	1.5	2	2.5	3
$y_1 = x^2 - 3$	–3	–2.75	–2	–0.75	1	3.25	6
$y_2 = \sqrt{x}$	0	0.707	1	1.224	1.414	1.581	1.732



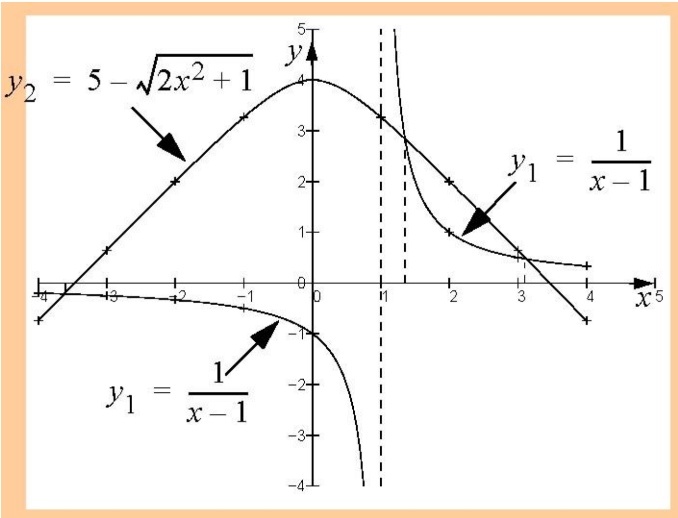
Solution is $x \approx 2.15$

b. $y_1 = \dfrac{1}{x-1}$, $y_2 = 5 - \sqrt{2x^2+1}$

Recognise that $y_1 = \dfrac{1}{x-1}$ is a hyperbola.

x	–4	–3	–2	–1	0	1	2	3	4
$y_1 = \dfrac{1}{x-1}$	–0.2	–0.25	–0.33	–0.5	–1	$\pm\infty$	1	0.5	0.33
$y_2 = 5 - \sqrt{2x^2+1}$	–0.74	0.64	2	3.27	4	3.27	2	0.64	–0.74

Note: There are three solutions.



Solutions are $x \approx -3.5, 1.3, 3.1$

Check that each solution satisfies the original equation.

Exercise 3.14

Use graphical methods to find approximate solutions to the following equations:

a. $0.5x = 0.3^x$

b. $x^2 + \sqrt[3]{x} = 2$

c. $3 - x = x^{1.5}$

d. $1 + \dfrac{1}{x-2} = \sqrt{x^2+1}$

e. $x^2 - 4x = \ln x$

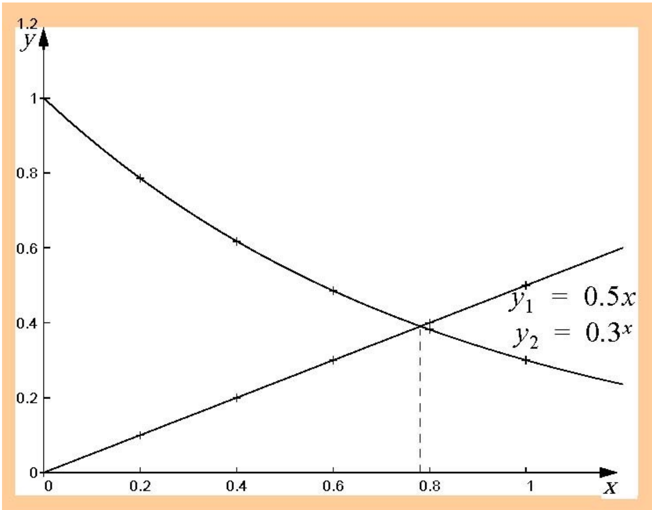
f. $2e^{-x} - \dfrac{1}{x+3} + 4 = 0$

(Note $\ln x = \log_e x$ – see Module 4 for logarithmic and exponential functions)

Solutions

a. $y_1 = 0.5x,\ y_2 = 0.3^x$

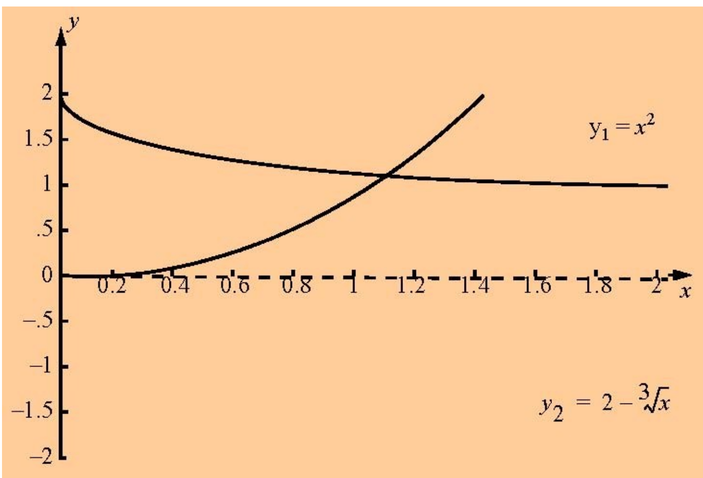
x	0	0.2	0.4	0.6	0.8	1
$y_1 = 0.5x$	0	0.1	0.2	0.3	0.4	0.5
$y_2 = 0.3^x$	1	0.79	0.62	0.49	0.38	0.3



Solution is approximately $x = 0.78$

b. $x^2 = 2 - \sqrt[3]{x},\ \therefore\ y_1 = x^2,\ y_2 = 2 - \sqrt[3]{x}$

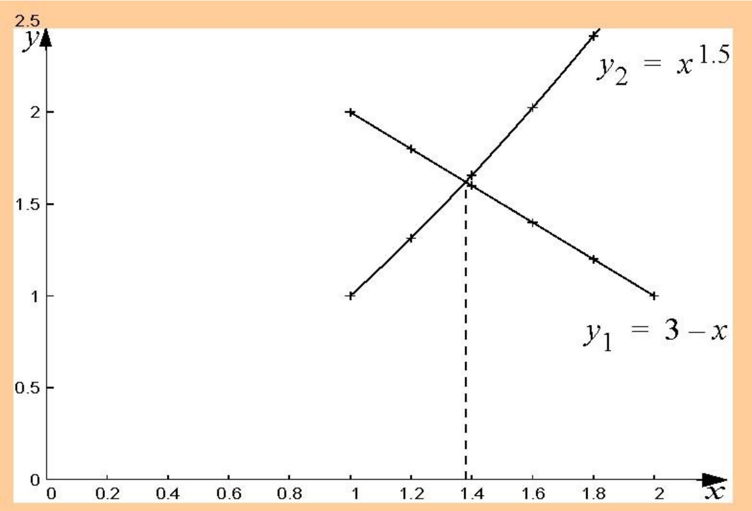
x	0	0.2	0.4	0.6	0.8	1	1.2	1.4
y_1	0	0.04	0.16	0.36	0.64	1	1.44	1.96
y_2	2	1.415	1.263	1.156	1.07	1	0.93	0.88



Solution is exactly $x = 1$

c. $y_1 = 3 - x,\ y_2 = x^{1.5}$

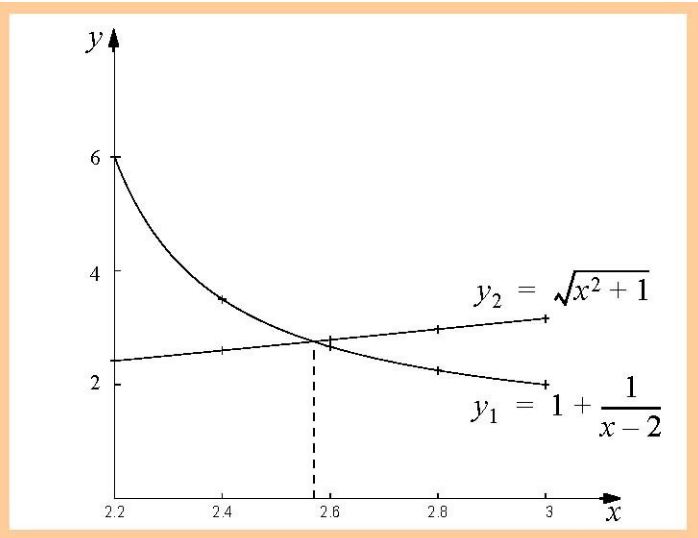
x	1	1.2	1.4	1.6	1.8	2
y_1	2	1.8	1.6	1.4	1.2	1
y_2	1	1.31	1.66	2.02	2.41	2.83



Solution is approximately $x = 1.4$

d. $y_1 = 1 + \dfrac{1}{x - 2},\ y_2 = \sqrt{x^2 + 1}$

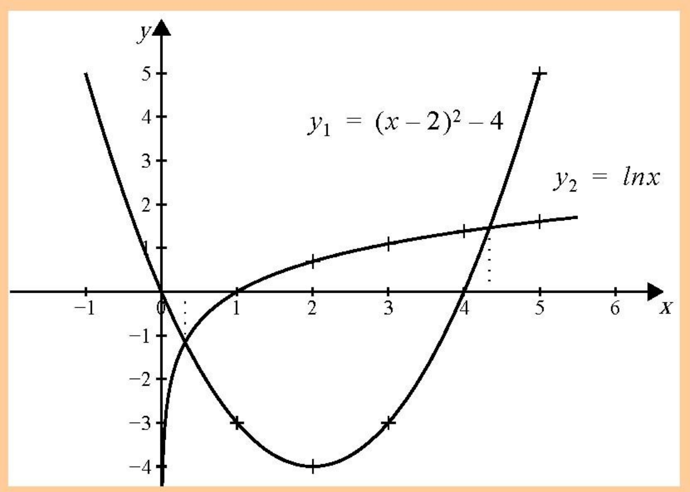
x	2.2	2.4	2.6	2.8	3
y_1	6	3.5	2.67	2.25	2
y_2	2.42	2.6	2.79	2.97	3.16



Solution is approximately $x = 2.53$

e. $y_1 = (x-2)^2 - 4, \quad y_2 = ln\ x$

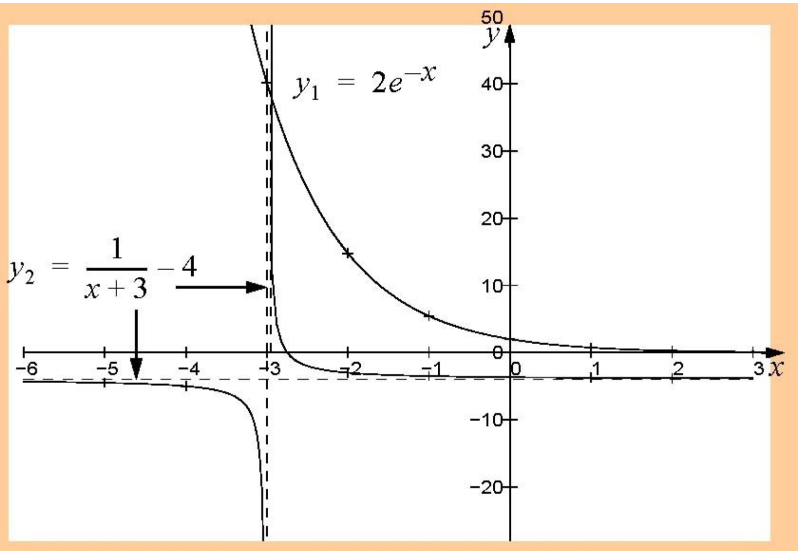
x	–10	1	2	3	4	5
y_1	50	–3	–4	–3	0	5
y_2	Und	0	0.69	1.1	1.38	1.61



Approximate solutions $x = 0.3$ and $x = 4.3$

f. $y_1 = 2e^{-x} \quad y_2 = \dfrac{1}{x+3} - 4$

x	–5	–4	–3	–2	–1	0	1	2
y_1	296.8	109.2	40.2	14.8	5.4	2	0.7	0.3
y_2	–4.5	–5	Undef	–3	–3.5	–3.7	–3.8	–3.8



Approximate solution $x = –2.9$

Module 3: Self assessment

Questions 3.1

1. Explain why $x^2 + y^2 = 1$ is not a function
2. If $f(x) = x^2 + 1$ and $g(x) = \sqrt{x - 1}$ evaluate
   i. $f(-1)$
   ii. $f(x + h)$
   iii. $f(g(x))$
   iv. domain of $g(x)$
3. Find the equation of the straight line passing through the points (4,3) and (6,7)
4. Sketch the graph of the function $y = x^2 - 5x + 6$
5. Complete the square in $x^2 + 3x + 1$
6. Sketch the function $y = \dfrac{1}{x - 2}$
7. Find the point of intersection of the three equations

   $3x - y + 2z = 3$
   $x + y + z = 3$
   $x - y + z = 1$
8. Indicate on a labelled sketch how you would graphically solve the equation

   $\sqrt{x} - \dfrac{3}{x - 2} = 4$ for $x \leq 3$
9. Write the equation of the circle with centre (4,2) passing through (–2,–1)

Questions 3.2

1. Which of the following relations are functions? Explain

   $y = 2x - 1$

   $y^2 = x$

   $y = x^2$

2. If $f(x) = 2x^2 + x + 1$ and $g(x) = x^2 - 1$ evaluate

   i. $f(-2)$

   ii. $f(x + a)$

   iii. $g(f(x))$

3. Find the equation of the straight line passing through the points (–2,–1) and (4,3)
4. What is the equation of the circle with centre (4,–2,) and passing through (2,2)?
5. Sketch the graph of the function $y = \dfrac{3}{x+1} - 1$
6. Complete the square in $x^2 + 4x + 1$
7. Find the point of intersection of the three equations

   $x + y + z = 0$

   $x - 3y + 2z = 1$

   $2x - 2y + z = -1$

8. Indicate on a labelled sketch how you would graphically solve the equation $x^2 - 2x = \dfrac{1}{x}$
9. Sketch the curve $y = x^3 - 2x^2 - x + 2$

Module 4 – Logarithms and related functions

Objectives

In this module you are required to be able to:

- demonstrate an understanding of the definition of the logarithmic function and the exponential function
- sketch the exponential and logarithmic functions
- solve real world problems including exponential and logarithmic functions
- apply logarithmic laws to simplify algebraic equations and formulae.

In previous sections we investigated expressions involving powers. In this module we will investigate this further by looking at an alternative way of representing powers – the logarithm.

The relationship between the logarithm and powers will be explored further by comparing the exponential and logarithmic functions. But firstly what is a logarithm?

4.1 Logarithms

Any positive number can be written as a power of 10.

For example, $100 = 10^2$

We call 10 the base and 2 the power, index or logarithm.

Also, $10\,000 = 10^4$

$0.001 = 10^{-3}$

Three can also be written as a power of ten. What power will it be?

$3 = 10^{\square}$

We know that $10^0 = 1$ and $10^1 = 10$

so the power of 10 that gives 3 will be between 0 and 1.

Depending on your type of calculator you should have a $\boxed{\log}$ button which allows us to determine logarithms (or powers) for base 10.

To determine which power of 10 gives 3, we need to find log 3.

On your calculator you should press 3 log

The calculator should read 0.4771212...

Hence $10^{0.4771212...} = 3$.

We can now write this as an equation

$$\log_{10} 3 = 0.4771212...$$

This would be read as 'log 3 to the base 10 equals 0.4771212...'

You may also see $\log_{10} 3$ written as log 3. The base 10 is inferred.

In general we can write the logarithmic equation as

$$y = \log_a x$$

which is read as 'y equals the log of x to the base a' or 'y equals log x to base a'.

Exercise 4.1

Give all answers to 4 decimal places.

1.
   a. log 4
   b. log 256
   c. log 0.5

2. What is x in each case?
   a. $2 = 10^x$
   b. $324 = 10^x$
   c. $0.3 = 10^x$

Solutions

1.
   a. 0.6021
   b. 2.4082
   c. –0.3010

2.
   a. 0.3010
   b. 2.5105
   c. –0.5229

The laws of logarithms

If $y = a^x$ then we can say that $\log_a y = x$. This is the basic relationship between powers and logarithms.

Examples

Example

Find $\log_3 9$

$$\begin{array}{ll} \text{Let} & \log_3 9 = x \\ \text{then} & 3x = 9 \\ & 3^x = 3^2 \\ & \therefore x = 2 \\ \text{That is,} & \log_3 9 = 2 \text{ since } 3^2 = 9. \end{array}$$

Example

Find $\log_2 \frac{1}{4}$

$$\begin{array}{ll} \text{Let} & \log_2 \frac{1}{4} = x \\ \text{then} & 2^x = \frac{1}{4} \\ & 2^x = \frac{1}{2^2} \\ & 2^x = 2^{-2} \\ & \therefore x = -2 \\ \text{That is,} & \log_2 \frac{1}{4} = -2 \text{ since } 2^{-2} = \frac{1}{4} \end{array}$$

Exercise 4.2

Find:

a. $\log_2 16$

b. $\log_5 125$

c. $\log_3 \frac{1}{9}$

d. $\log_4 64$

e. $\log_e e^2$

f. $\log_3 \sqrt{3}$

g. $\log_{10} \frac{1}{1000}$

h. $\log_3 243$

Solutions

a. $\log_2 16 = 4$

b. $\log_5 125 = 3$

c. $\log_3 \frac{1}{9} = \log_3 3^{-2} = -2$

d. $\log_4 64 = 3$

e. $\log_e e^2 = 2$

f. $\log_3 \sqrt{3} = \log_3 3^{\frac{1}{2}} = \frac{1}{2}$

g. $\log_{10} \frac{1}{1000} = \log_{10} 10^{-3} = -3$

h. $\log_3 243 = 5$

In section 1.4 we looked at a series of rules for combining powers. We will now look at a similar series of rules for combining logarithms.

Rule 1. Logarithm of a product.

$$\log_a (m \times n) = \log_a m - \log_a n$$

Example

$$\begin{aligned} \log_{10} 6 &= 0.778 \\ \text{now } \log_{10} 6 &= \log_{10} (2 \times 3) \\ &= \log_{10} 2 + \log_{10} 3 \\ &= 0.301 + 0.477 \\ &= 0.778 \end{aligned}$$

We can use this rule to combine two logarithms that are added together.

Example

$$\begin{aligned} \log_3 2 + \log_3 x &= \log_3 (2 \times x) \\ &= \log_3 2x \end{aligned}$$

Rule 2. Logarithm of a quotient.

$$\log_a \frac{m}{n} = \log_a m - \log_a n$$

Example

$$\begin{aligned} \log_{10} 4 &= 0.602 \\ \text{now } \log_{10} 4 &= \log_{10} \frac{8}{2} \\ &= \log_{10} 8 - \log_{10} 2 \\ &= 0.903 - 0.301 \\ &= 0.602 \end{aligned}$$

We can also use this rule to combine two logarithms that are subtracted.

$$\log_3 2 - \log_3 x = \log_3 \frac{2}{x}$$

Rule 3. Logarithm of a power.

$$\log_a x^p = p \log_a x$$

Example

$\log_{10} 9 = 0.954$ $\quad$ $\log_{10} 9$ can be written as $\log_{10} 3^2$

$\log_{10} 3^2 = 2 \log_{10} 3 = 2 \times 0.477 = 0.954$

Rule 4. Logarithm of 1 to any base.

$$\log_a 1 = 0$$

The log of 1 always equals zero!

Rule 5. Logarithm of number to the same base.

$$\log_a a = 1$$

Example

$\log_{10} 10 = 1$ $\quad$ since $10^1 = 10$

We can now apply these rules to a variety of situations.

Example

Simplify $3\log 2 + \log 125$ Recall that, if no base is stated it is understood to be base 10

$$\begin{aligned} 3\log 2 + \log 125 &= \log 2^3 + \log 125 \\ &= \log 8 + \log 125 \\ &= \log(8 \times 125) \\ &= \log 1000 \\ &= 3 \\ \text{Since } 10^3 &= 1000 \end{aligned}$$

Example

Simplify $\log xy^3 - \log x^2y + \log x$

$$\begin{aligned} \log xy^3 - \log x^2y + \log x &= \log \frac{xy^3}{x^2y} + \log x \\ &= \log\left(\frac{xy^3}{x^2y} \times x\right) \\ &= \log y^2 \end{aligned}$$

Exercise 4.3

1. Find x, given that

   a. $\log_4 x = 2$

   b. $\log_x 49 = 2$

   c. $\log_5 625 = x$

   d. $\log_x 343 = 3$

2. Simplify

   a. $\log_2 16 + \log_2 8$

   b. $\log_6 2 + \log_6 3$

   c. $\log_3 54 - \log_3 18$

   d. $\log_2 18 - 2\log_2 3$

   e. $\log 25 + \log 3 - \log 5$

   f. $2\log_5 5 - \log_5 10 + \log_5 \frac{1}{5}$

3. Simplify

   a. $\dfrac{\log x^3}{\log x}$

   b. $\log_3 (3^x)$

   c. $\dfrac{\log_{10} 36}{\log_{10} 6}$

   d. $\dfrac{\log x}{\log \sqrt{x}}$

Solutions

1. a
$$\begin{aligned} \log_4 x &= 2 \\ x &= 4^2 \\ x &= 16 \end{aligned}$$

   b.
$$\begin{aligned} \log_x 49 &= 2 \\ 49 &= x^2 \\ x &= 7 \end{aligned}$$
(we cannot have a negative base so –7 is ignored)

   c.
$$\begin{aligned} \log_5 625 &= x \\ 625 &= 5^x \\ 5^4 &= 5^x \\ x &= 4 \end{aligned}$$

   d.
$$\begin{aligned} \log_x 343 &= 3 \\ 343 &= x^3 \\ 7^3 &= x^3 \\ \therefore x &= 7 \end{aligned}$$

   Another way to do this would be to use the power rules.

$$\begin{aligned} \log_x 343 &= 3 \\ 343 &= x^3 \\ (343)^{\frac{1}{3}} &= (x^3)^{\frac{1}{3}} \\ 7 &= x \end{aligned}$$

2. a
$$\begin{aligned} \log_2 16 + \log_2 8 &= \log_2 16 \times 8 \\ &= \log_2 128 \\ &= 7 \end{aligned}$$

   b.
$$\begin{aligned} \log_6 2 + \log_6 3 &= \log_6 2 \times 3 \\ &= \log_6 6 \\ &= 1 \end{aligned}$$

c.

$$\begin{aligned}\log_3 54 - \log_3 18 &= \log_3 \frac{54}{18}\\ &= \log_3 3\\ &= 1\end{aligned}$$

d.

$$\begin{aligned}\log_2 18 - 2\log_2 3 &= \log_2 18 - \log_2 3^2\\ &= \log_2 18 - \log_2 9\\ &= \log_2 \frac{18}{9}\\ &= \log_2 2\\ &= 1\end{aligned}$$

e.

$$\begin{aligned}\log 25 + \log 3 - \log 5 &= \log(25 \times 3) - \log 5\\ &= \log 75 - \log 5\\ &= \log \frac{75}{5}\\ &= \log 15\end{aligned}$$

f.

$$\begin{aligned}2\log_5 5 - \log_5 10 + \log_5 \frac{1}{5} &= \log_5 5^2 - \log_5 10 + \log_5 \frac{1}{5}\\ &= \log_5 25 - \log_5 10 + \log_5 \frac{1}{5}\\ &= \log_5 \frac{25 \times \frac{1}{5}}{10}\\ &= \log_5 \frac{1}{2}\\ &= \log_5 2^{-1}\\ &= -\log_5 2\end{aligned}$$

3. a

$$\begin{aligned}\frac{\log x^3}{\log x} &= \frac{3\log x}{\log x}\\ &= 3\end{aligned}$$

b.

$$\begin{aligned}\log_3 3^x &= x\log_3 3\\ &= x \times 1\\ &= x\end{aligned}$$

c.

$$\begin{aligned}\frac{\log_{10} 36}{\log_{10} 6} &= \frac{\log_{10} 6^2}{\log_{10} 6}\\ &= \frac{2\log_{10} 6}{\log_{10} 6}\\ &= 2\end{aligned}$$

d.
$$\frac{\log x}{\log\sqrt{x}} = \frac{\log x}{\log x^{\frac{1}{2}}}$$
$$= \frac{\log x}{\frac{1}{2}\log x}$$
$$= \frac{1}{\frac{1}{2}}$$
$$= 2$$

4.2 Equations involving logarithms

In any equation, if the unknown is in the power (exponent) then logarithms can be used to solve the equation. Practical applications of logarithms will be covered in section 4.5.

Examples

Example

Solve $2^x = 120$

$$2^x = 120$$
$$\log 2^x = \log 120 \quad \text{take the logarithm of both sides}$$
$$x \log 2 = \log 120 \quad \text{using the logarithm of a power rule}$$
$$\frac{x\log 2}{\log 2} = \frac{\log 120}{\log 2}$$
$$x = \frac{\log 120}{\log 2}$$
$$= \frac{2.079}{0.301} \quad \text{on your calculator use the log key}$$
$$= 6.907$$

Example

Solve $100 = 50(10)^{3t}$

$$
\begin{aligned}
100 &= 50(10)^{3t} \\
\frac{100}{50} &= 10^{3t} \\
2 &= 10^{3t} \\
\log_{10} 2 &= \log 10^{3t} \\
\log_{10} 2 &= 3t \log 10 \\
\log_{10} 2 &= 3t \qquad \text{recall } \log_{10} 10 = 1 \\
\frac{\log_{10} 2}{3} &= t \\
t &= \frac{0.3010}{3} \qquad \text{on your calculator use the log key} \\
t &= 0.1
\end{aligned}
$$

Exercise 4.4

Solve the following equations giving answers to 3 decimal places.

a. $9^x = 25$

b. $7^{x+1} = 11$

c. $300 = 20\,(10)^{4t}$

d. $40 = 100\,(10)^{2t}$

Solutions

a. $9^x = 25$

$$
\begin{aligned}
\log 9^x &= \log 25 \\
x \log 9 &= \log 25 \\
x &= \frac{\log 25}{\log 9} \\
&= \frac{1.39794}{0.95424} \\
&= 1.465
\end{aligned}
$$

b. $7^{x+1} = 11$

$$
\begin{aligned}
\log 7^{x+1} &= \log 11 \\
(x+1)\log 7 &= \log 11 \\
x \log 7 + \log 7 &= \log 11 \\
x \log 7 &= \log 11 - \log 7 \\
x &= \frac{\log 11 - \log 7}{\log 7} \\
&= \frac{0.19629}{0.84510} \\
&= 0.232
\end{aligned}
$$

c. $300 = 20\,(10)^{4t}$

$$
\begin{aligned}
\frac{300}{20} &= (10)^{4t} \\
\log 15 &= 4t \log 10 \\
\log 15 &= 4t \\
\frac{\log 15}{4} &= t \\
t &= 0.294
\end{aligned}
$$

d. $40 = 100\,(10)^{2t}$

$$
\begin{aligned}
\frac{40}{100} &= 10^{2t} \\
\log 0.4 &= 2t \log 10 \\
\log 0.4 &= 2t \\
\frac{\log 0.4}{2} &= t \\
t &= -0.199
\end{aligned}
$$

We can use these same ideas to change the subject of a formula where one of the variables is a power.

Examples

Example

Make y the subject of $x = 5^y$

From the basic relationship between powers and logarithms $\log_5 x = y$ or $x = 5^y$.

$$\begin{aligned} \log x &= \log 5^y && \text{take the logarithm of both sides} \\ \log x &= y \log 5 \\ \frac{\log x}{\log 5} &= y \\ y = &= \frac{\log x}{\log 5} \end{aligned}$$

Example

Make y the subject of $x = \log y + 1$

$$\begin{aligned} x &= \log y + 1 \\ x - 1 &= \log y \\ 10^{x-1} &= y \\ y &= 10^{x-1} && \text{recall the basic relationship between logarithms and powers, if } \log_a x = y \text{ then } a^y = x \end{aligned}$$

Exercise 4.5

Use logarithms to make y the subject of the following equations.

a. $x = 6^y$

b. $x = (2 + b)^y$

c. $x = (a + 2b)^{6y-1}$

d. $x = \dfrac{ab^{2y}}{c^y}$

Solutions

a.

$$\begin{aligned} \log x &= \log 6y \\ \log x &= y \log 6 \\ \frac{\log x}{\log 6} &= y \text{ or } y = \log_6 x \end{aligned}$$

b.

$$\begin{aligned}
\log x &= \log (2 + b)^y \\
\log x &= y \log (2 + b) \\
\frac{\log x}{\log(2 + b)} &= y
\end{aligned}$$

c.

$$\begin{aligned}
\log x &= \log (a + 2b)^{6y-1} \\
\log x &= (6y - 1) \log (a + 2b) \\
6y - 1 &= \frac{\log x}{\log(a + 2b)} \\
6y &= \frac{\log x}{\log(a + 2b)} + 1 \\
y &= \frac{1}{6}\left(\frac{\log x}{\log(a + 2b)} + 1\right)
\end{aligned}$$

d.

$$\begin{aligned}
\log x &= \log\left(\frac{ab^{2y}}{c^y}\right) \\
\log x &= \log (ab^{2y}) - \log (c^y) \\
\log x &= \log (a) + \log (b^{2y}) - \log (c^y) \\
\log x &= \log a + 2y \log b - y \log c \\
\log x - \log a &= 2y \log b - y \log c \\
\log x - \log a &= y (2 \log b - \log c) \\
\frac{\log x - \log a}{2\log b - \log c} &= y
\end{aligned}$$

4.3 Exponential functions

We know now that logarithms and powers are closely related. Lets look at the exponential function more closely by examining bank interest rates. This is just one example of where the exponential function occurs in the studies of business, science and engineering.

A certain bank claims to pay 10% p.a. interest on a fixed deposit. This means that for each year of the investment the value of the deposit grows by the same rate of 10%. Complete the table below.

	Value of investment	**Calculations**
Initial investment	\$1 000	\$1 000
After 1 year	\$1 000 + 10% of \$1 000 = \$1 100	$= 1\,000 + 0.1 \times 1\,000$ $= 1\,000\,(1 + 0.1)$ $= 1\,000 \times 1.1$
After 2 years	\$1 100 + 10% of \$1 100 = \$ ______	$= (1\,000 \times 1.1) + 0.1 \times (1\,000 \times 1.1)$ $= (1\,000 \times 1.1)\,[1 + 0.1]$ $= 1\,000 \times 1.1 \times 1.1$ $= 1\,000 \times (1.1)^2$
After 3 years	________ + 10% of _____ = _______	$= 1\,000 \times (1.1)^2 + 0.1 \times (1\,000 \times (1.1)^2)$ $= 1\,000 \times (1.1)^2\,[1 + 0.1]$ $= 1\,000 \times (1.1)^2 \times 1.1$ $= 1\,000 \times (1.1)^3$
After 4 years	________ + 10% of _____ = _______	$= 1\,000 \times (1.1)^3 + 0.1 \times (1\,000 \times (1.1)^3)$ $= 1\,000 \times (1.1)^3\,[1 + 0.1]$ $= 1\,000 \times (1.1)^3 \times 1.1$ $= 1\,000 \times (1.1)^4$

You should have found the value of the investment to be \$1464.10 after 4 years.

You should have noticed that the value of the investment increased by a factor of 1.1 each year. The value of the investment after 1 year is 1.1 times that initially invested (\$1000).

After 2 years it is 1.12 times that initially invested and after 3 years 1.13 times that initially invested.

Using this pattern what will be the value of the investment after 10 years?

After 10 years of investment the value will be \$2 593.74.

That is $1.1^{10} \times 1\,000$.

Write a formula for this pattern.

You should have written something like

$$V = 1\,000 \times 1.1^t$$

where V equals the value of the investment after t years.

This is an exponential equation as it represents exponential growth.

What is the domain of this exponential function?

Did you say t values greater than or equal to 0, or $t \geq 0$, or $[0,\infty)$?

What about the range?

The range would be values of V greater than or equal to 1 000, or

$V \geq 1\ 000$, or $[1\ 000,\infty)$.

If we look more closely at the above example we can derive a formula for interest earned in this way. That is, where the interest is added to the amount invested (the principal) and the next interest payment is calculated on this new amount.

We call this **compound interest** because we are receiving interest on interest.

We can write 1.1 as $1 + 0.1 = 1 + \dfrac{10}{100}$

$$V = 1000\left(1 + \frac{10}{100}\right)^t$$

If we now write this in general terms

$$A = P\left(1 + \frac{r}{100}\right)^n$$

where A = the amount after n periods

P = the principal invested (initial amount)

r = the interest rate per period of time

n = the number of periods.

For our example the period of time was one year but this could be one month, or every three months (quarterly) or even every six months.

Examples

Example

If James invested \$100 in an account paying 12% per annum paid quarterly:

a. how much is in the account after 5 years?

b. how long should James invest the money to have \$800?

Use the compound interest formula

$$A = P\left(1 + \frac{r}{100}\right)^n$$

a. 12% quarterly is $\frac{12\%}{4}$ = 3% per quarter.

5 years is 5 × 4 = 20 quarters

$$A = 100\left(1 + \frac{3}{100}\right)^{20}$$

$$= 100\,(1.03)^{20}$$

$$= 100 \times 1.8061112$$

$$= \$180.61$$

∴ after 5 years James would have \$180.61

b. This time we know A the amount we will have at the end of the time but we don't know how many terms there will be.

$$A = P\left(1 + \frac{r}{100}\right)^{n}$$

$$800 = 100\left(1 + \frac{3}{100}\right)^{n}$$

$$800 = 100(1.03)^{n}$$

$$\frac{800}{100} = (1.03)^{n}$$

$$8 = (1.03)^{n}$$

When the unknown is in the index we will use logarithms to solve the equation.

$\log 8 = \log (1.03)^n$ take the logarithm of both sides

$\log 8 = n \log (1.03)$ use the laws of logarithms to simplify

$$n = \frac{\log 8}{\log(1.03)}$$

$$= \frac{0.90309}{0.01284}$$

$$n = 70.3$$

This means that we must leave the \$100 invested for approximately 71 quarters or 17 years and 9 months.

Example

The taxation department allows 30% p.a. for depreciation of computer equipment.

If Sandra has a computer valued at \$4 500, what will be its value after 3 years?

This time instead of the amount increasing each year it is decreasing by the given rate.

$$\text{That is } A = P\left(1 + \frac{r}{100}\right)^n$$

$$\text{So, } A = 4500\left(1 - \frac{30}{100}\right)^3$$

$$= 4500\ (1 - 0.3)^3$$

$$= 4500\ (0.7)^3$$

$$= 4500 \times 0.343$$

$$= 1543.50$$

Sandra's computer is worth \$1 543.50 after three years.

We have seen so far that exponential functions occur in man made situations but exponential growth and decay are also a part of nature.

Exponential growth is said to occur when the **rate of growth** of a population is proportional to its size.

Example

In a certain bacterial culture the number (N) of bacteria present is given by the formula

$$N = 800(2)^{3t}$$

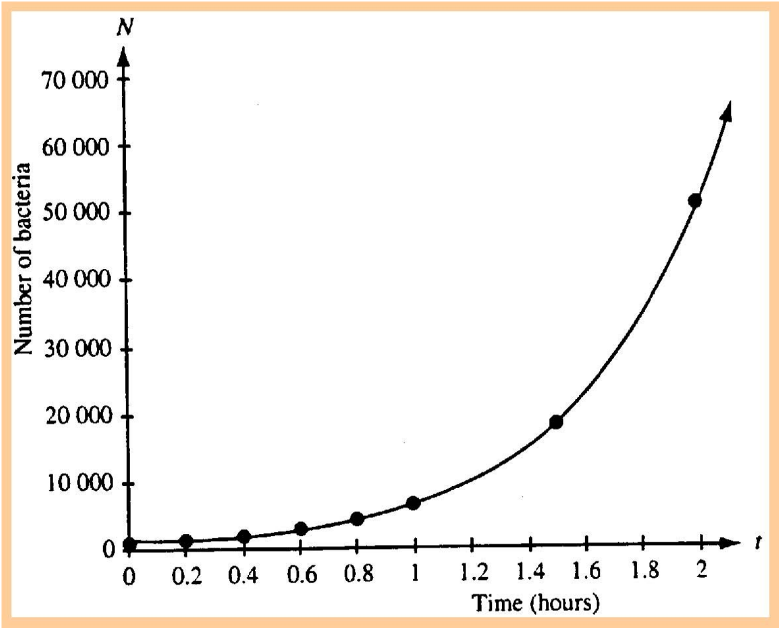
where t is the time (hours) that the bacterial culture has been growing.

a. Sketch the graph of the function
b. How many bacteria are present at the start?
c. How many bacteria will be present after 4 hours?
d. How long will it take for the number of bacteria to double?

Solutions

a.

t	0	0.2	0.4	0.6	0.8	1	1.5	2
$N = 800(2)^{3t}$	800	1213	1838	2786	4222	6400	18102	51200



We call this is an **exponential growth function**.

b. The number of bacteria present at the start is found by substituting $t = 0$.

So, $N = 800(2)^{3 \times 0}$

$= 800 \times (2)^0$

$= 800$

There are 800 bacteria present at the start.

c. After 4 hours the value of t is 4

So, $N = 800(2)^{3 \times 4}$

$= 800(2)^{12}$

$= 800 \times 4\,096$

$= 3\,276\,800$

After 4 hours there are 3 276 800 bacteria present.

d. When the number of bacteria have doubled there will be $2 \times 800 = 1600$ bacteria present.

$$\begin{aligned} \text{So, } 1600 &= 800\,(2)^{3t} \\ \frac{1600}{800} &= (2)^{3t} \\ 2 &= 2^{3t} && \text{since the bases are the same the powers must be equal} \\ 1 &= 3t \\ t &= \frac{1}{3} \end{aligned}$$

It takes ⅓ of 1 hour or 20 minutes for the number of bacteria to double.

Exercise 4.6

1. A single cholera bacterium divides every half hour to produce two complete bacteria. If we start with a colony of 5000 bacteria, then after t hours we will have 5000×2^{2t} bacteria. How long will it take to reach 1 000 000 bacteria?
2. The pressure of the atmosphere (P) decreases with the height (h) above sea level according to the law

$$P = P_o 10^{-bh} \qquad \text{where } P_o \text{ is the initial pressure and } b \text{ is a constant}$$

If the pressure at sea level is 76 cm and at 4 km above sea level the pressure is 34.15 cm calculate

a. the value of b

b. the pressure at 10 km above sea level.

Solutions

1.
$$\begin{aligned} 1\,000\,000 &= 5000 \times 2^{2t} \\ \frac{1000000}{5000} &= 2^{2t} \\ 200 &= 2^{2t} \\ t &= \frac{\log 200}{2\log 2} \\ &= \frac{2.3010}{0.6021} \\ &\approx 3.82 \end{aligned}$$

It takes approximately 3.82 hours to reach 1 000 000 bacteria.

2. a. $P = P_o 10^{-bh}$

The pressure at sea level is 76 cm, i.e. $P_o = 76$.

$P = 76 \times 10^{-bh}$

when $h = 4$ $P = 34.15$

$$\begin{aligned}
34.15 &= 76 \times 10^{-bh} \\
\frac{34.15}{76} &= 10^{-4b} \\
\log\left(\frac{34.15}{76}\right) &= 4b\log 10 \\
b &= \frac{\log\left(\frac{34.15}{76}\right)}{-4} \qquad \log 10 = 1 \\
&= \frac{-0.3474}{-4} \\
&= 0.087 \\
P &= 76 \times 10^{-0.087h}
\end{aligned}$$

b. When $h = 10$

$$\begin{aligned}
P &= 76 \times 10^{-0.087 \times 10} \\
&= 76 \times 0.1349 \\
&= 10.25 \text{ cm}
\end{aligned}$$

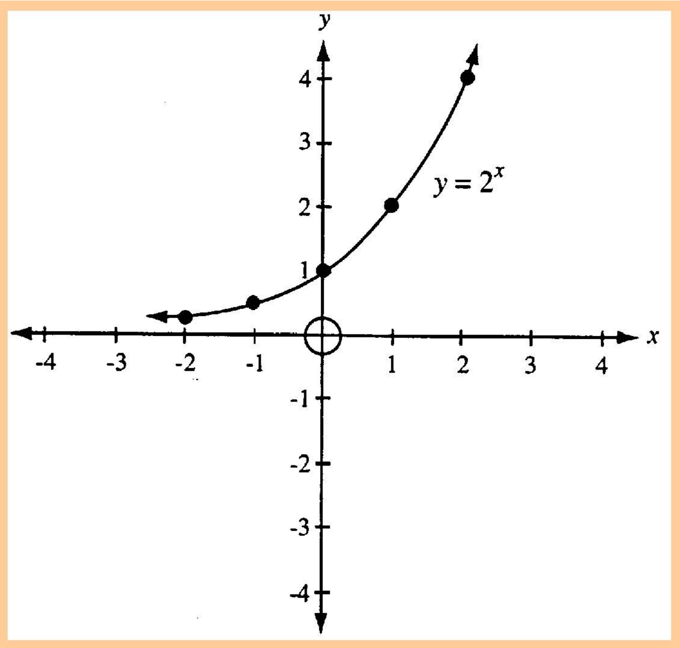
At 10 km above sea level the pressure is 10.25 cm.

4.4 Logarithmic functions

The logarithmic curve is closely related to the exponential curve.

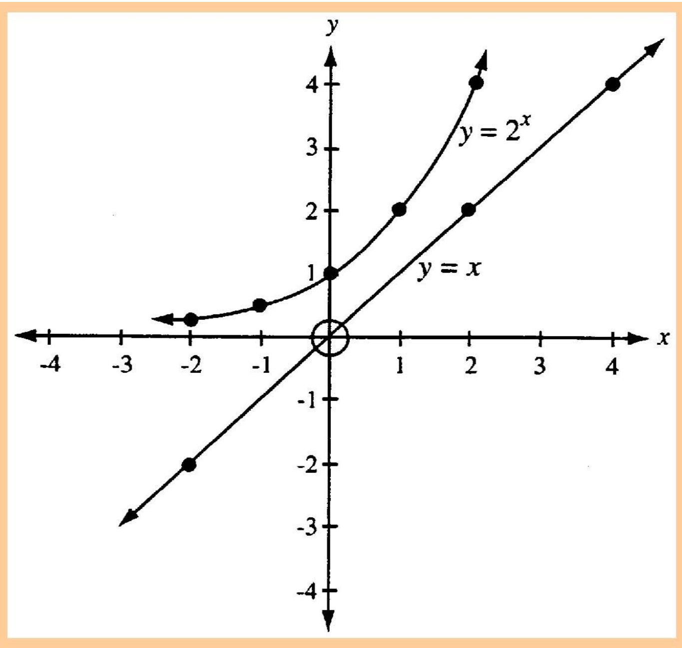
Consider $y = 2^x$

Sketch over the domain $-2 \le x \le 2$ and **use the same scale for both axes**.

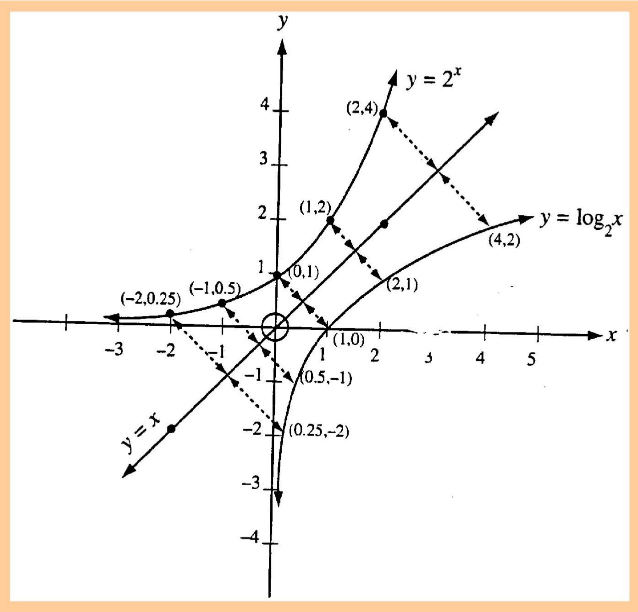


On the same cartesian plane, let's draw the line $y = x$.

x	-2	0	2
$y = x$	-2	0	2



Now let us reflect the curve $y = 2^x$, so that the line $y = x$ acts like a mirror. We call this process finding the **inverse** of the function.



The inverse of an exponential function is called a **logarithmic function**. The curve above is

$$y = \log_2 x$$

which is read as '*y* equals the log of *x* to the base 2'.

Note that the log of a negative number does not exist. As *x* approaches zero, the curve approaches negative infinity and does not touch the *y*-axis.

Also note that the curve flattens out for large values of *x* – although it never becomes perfectly level.

Examples

Draw the inverses of the following functions. To do this

- graph the exponential functions for $-2 \le x \le 2$
- reflect all points in the line $y = x$

a. $y = 3^x$

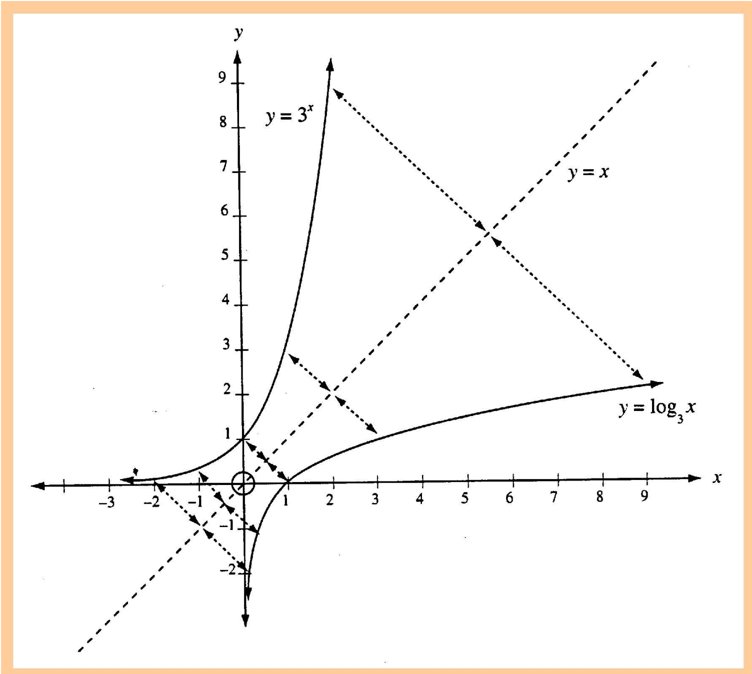
b. $y = 4^x$

c. $y = e^x$

Solutions

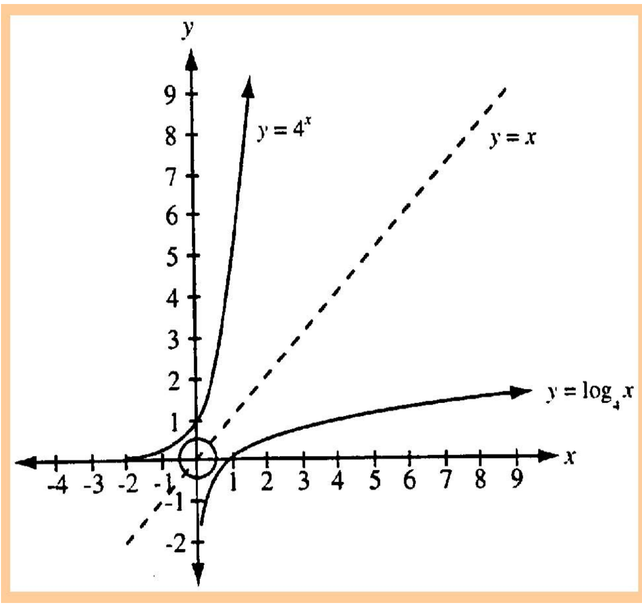
a.

x	-2	-1	0	1	2
$y = 3^x$	0.1	0.3	1	3	9



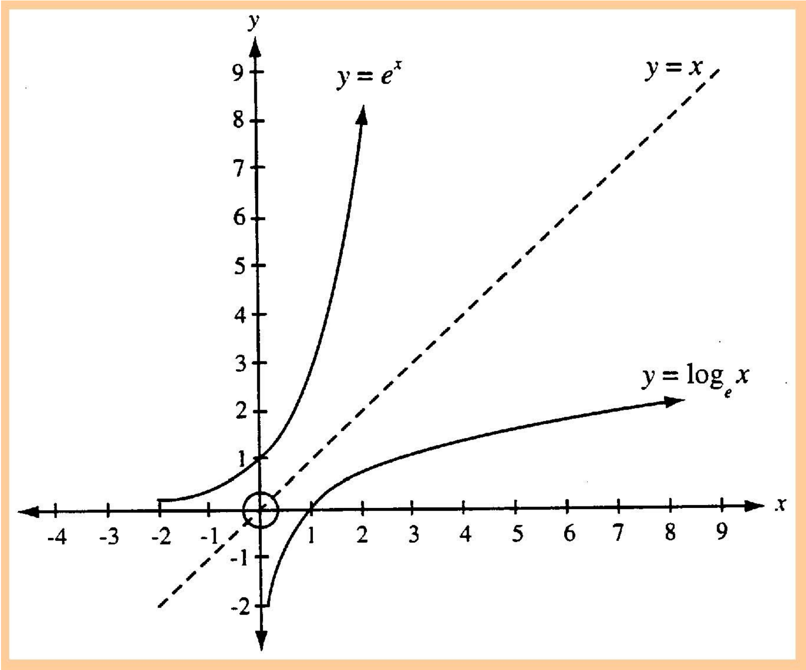
b.

x	-2	-1	0	1	2
$y = 4^x$	0.06	0.25	1	4	16



c.

x	-2	-1	0	1	2
$y = e^x$	0.1	0.4	1	2.7	7.4



Let's look more closely at the last graph that you drew.

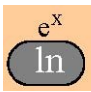
$y = \log_e x$

We use this logarithm base e so often that it has a special notation.

$y = \ln x$ where ln = log base e

and we call it the **natural logarithm**.

All the rules for logarithms covered in section 4.1 hold for the natural logarithm.

Your calculator should have a 

button so you can evaluate expressions involving logarithms to the base e.

Exercise 4.7

1. Evaluate
   a. e^7
   b. $\log_e 4$
   c. $\log_{10} 2$
   d. $\ln 24$
   e. $\log 56$
2. What is x in each case?
   a. $4 = e^x$
   b. $10 = e^x$
   c. $32 = 10^x$
   d. $e = 10^x$

Solutions

1.
   a. 1 096.6332
   b. 1.3863
   c. 0.3010
   d. 3.1781
   e. 1.7482
2.
   a. 1.3863
   b. 2.3026
   c. 1.5051
   d. $\log e = 0.4343$
      Find e first, then find the log of this value.

We can also use the natural logarithm in a more practical situation.

Example

Radioactive material is known to decay at a rate given by the equation

$$M = M_o e^{-kt}$$

where M = mass of radioactive material remaining

M_o = initial mass

k = decay constant

t = time (years)

a. If 10 kg reduces to 6 kg in 3 months, find the time for the material to decay to a mass of 2 kg.

b. If **half-life** is the time taken for half of the amount of a material to decay, find the half life of this material.

Solutions

a. If there is 10 kg of material at the start then M_o is equal to 10 kg. Since time is measured in years 3 months is 0.25 years. This is the time taken for the material to decay to 6 kg.

$$6 = 10e^{-k \times 0.25}$$

$$\frac{6}{10} = e^{-k \times 0.25}$$

$$\ln(0.6) = \ln_e^{-0.25k}$$

$$\ln(0.6) = -0.25k \times \ln e \qquad \text{recall } \log_e e = 1$$

$$k = \frac{\ln(0.6)}{-0.25}$$

$$= 2.043302495 \qquad \text{it is a good idea to keep this value in the memory for later use}$$

Since we now know the decay constant it is possible to find the time taken for the material to reduce to 2 kg.

$$2 = 10e^{-2.043302495 \times t}$$

$$0.2 = e^{-2.043302495 \times t}$$

$$\ln 0.2 = -2.043302495t \ln e$$

$$t = \frac{\ln 0.2}{-2.0433025}$$

$$= 0.788$$

It takes 0.788 years to reduce to 2 kg.

b. If the original mass was 10 kg the half life is the time taken for the mass to decay to 5 kg.

$$
\begin{aligned}
5 &= 10e^{-2.043302495 \times t} \\
t &= \frac{\ln 0.5}{-2.0433025} \\
&= 0.339
\end{aligned}
$$

It takes approximately 0.339 years to reduce to half the original mass.

4.5 Further applications of logarithms and exponentials

Exercise 4.8

1.

a. Over a period of time the population of an old mining town was decreasing at a rate of 7% per year. If the original population was 12 600, what was the population 5 years later?

b. In how many years will \$600 amount to \$1000 invested at 6% p.a. compound interest?

c. An author's library valued at \$2 400 depreciated at the rate of 8% p.a. Find the value of the library after 4 years?

d. \$2000 is invested in a term deposit paying 10% p.a. paid quarterly. Find the:

i. amount after 3 years and

ii. time for the investment to be \$5 000.

2.

a. The number of bacteria (B) in a medium with ample food after t minutes is given by

$$B = 1000e^{0.03t}$$

Find the

i. number of bacteria after 7 minutes

ii. time for the number of bacteria to reach 2000

b. In radioactive materials, the rate of decay is given by

$M = M_o e^{-kt}$ Where M is the mass in grams
M_o is the initial mass
t is the time in years
k is the decay constant

If it takes 2 years for a piece of Uranium to decrease in mass from 10 g to 8 g, find the mass after 6 years.

Solutions

1. a. $P = 12600\left(1 - \frac{7}{100}\right)^5$

$= 12600\ (0.93)^5$

$= 8\ 765.67$

After 5 years the population would be approximately 8 766 people.

b.
$$\begin{aligned}1000 &= 600\left(1+\frac{6}{100}\right)^n\\ \frac{1000}{600} &= (1.06)^n\\ \log\left(\frac{10}{6}\right) &= n\log 1.06\\ n &= \frac{\log\left(\frac{10}{6}\right)}{\log 1.06}\\ &= 8.766693\end{aligned}$$

After approximately 9 years there will be \$1000.

c. $V = 2400\left(1-\frac{8}{100}\right)^4$

$= 2400\,(0.92)^4$

$= 1719.3431$

The value of the library after 4 years will be approximately \$1 719.

d. 10% paid quarterly is $\frac{10}{4}$ % each quarter.

i. 3 years is 12 quarters

The amount after 3 years is given by

$A = 2000\left(1+\frac{2.5}{100}\right)^{12}$

$= 2000\,(1.025)^{12}$

$= 2689.78$

After 3 years the deposit is \$2 689.78

ii. $5000 = 2000\,(1.025)^n$

$2.5 = (1.025)^n$

$\log 2.5 = n \log 1.025$

$n = \frac{\log 25}{\log 1.025}$

$= 37.10789$

After 38 quarters ($9\frac{1}{2}$ years) the deposit will be \$5 000.

2. a. i. When

$t = 7$

$B = 1\ 000e^{0.03 \times 7}$

$= 1\ 000e^{0.21}$

$= 1\ 233.678$

$\therefore$ after 7 minutes there are approximately 1234 bacteria.

ii. When

$B = 2\ 000$

$2000 = 1\ 000e^{0.03t}$

$2 = e^{0.03t}$

$\ln 2 = 0.03t$

$t = \dfrac{\ln 2}{0.03}$

$= 23.1049$

After about 24 minutes there are 2000 bacteria.

b. The initial mass was 10 g.

$\therefore M = 10e^{-kt}$

After 2 years the mass is 8 g

$8 = 10e^{-2k}$

$\frac{8}{10} = e^{-2k}$

$\ln 0.8 = -2k$

$k = \frac{\ln 0.8}{-2}$

$= 0.1115718$

After 6 years the mass will be

$M = 10e^{-0.1115718 \times 6}$

$= 10e^{-0.6694308}$

$= 5.12$

After 6 years the mass of Uranium left is 5.12 grams.

Some important points about e^x and $ln\ x$

i. $x = e^{ln\,x}$, $x = ln(e^x)$

ii. $ln\ e = 1$ (i.e. $log_e e = 1$)

iii. e^x can be expressed as the sum of an infinite series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

where $n! = n(n-1)(n-2) \ldots (3)(2)(1)$ and $0! = 1! = 1$

iv. e^x can be expressed as the value of the limit of the function

$$e^x = 1 \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

v. e^x is the only function which remains unaltered when differentiated or integrated

Examples

Example

Radioactive material is known to decay to inert material at a rate given by the exponential equation:

$$x = x_0 e^{-kt}$$

where x = mass of radioactive material remaining at time t

x_0 = initial mass

k = decay constant

t = time (years)

Find the time required for 10 kilograms of radioactive material with a decay constant of 1.3 to decay to a mass of 6 kilograms.

Solution

$$x = x_0 e^{-kt}$$

$$x = 10e^{-1.3t}$$

$$\frac{x}{10} = e^{-1.3t}$$

Taking natural log of each side gives

$$\ln\left(\frac{x}{10}\right) = \ln(e^{-1.3t}) = -1.3t\,\ln e = -1.3t \;\{\ln e = 1\}$$

$$t = -\frac{\ln\left(\frac{x}{10}\right)}{1.3}$$

When $x = 6$, $t = -\dfrac{\ln 0.6}{1.3} = -\dfrac{-0.5108}{1.3} = 0.393$ years

Example

The world's population is known to increase at a rate given by:

$$P = Ae^{Bt}$$

where P = population at time t

t = time (years)

A = population at time zero (in millions)

B = constant

If the population was 2500 million in 1945 and 4000 million in 1970, find B and estimate the population in 1985 and 2000.

Solutions

Let time be zero in 1945.

$$A = 2500$$

Given P = 4000 when t = (1970–1945) = 25

$$4000 = Ae^{25B} = 2500e^{25B}$$

$$e^{25B} = \frac{4000}{2500} = 1.6$$

Taking natural log of each side gives

$$\ln (e^{25B}) = \ln 1.6$$

$$25B \ln e = \ln 1.6$$

$$25B = \ln 1.6$$

$$B = \frac{\ln 1.6}{25} = \frac{0.47}{25} = 0.01880$$

$$P = 2500e^{0.0188t}$$

In 1985, t = (1985–1945) = 40

When $t = 40$, $P = 2500\ e^{0.0188 \times 40} = 2500\ e^{0.752}$

$$= 2500 \times 2.121 = 5303 \text{ million}$$

In 2000, t = (2000–1945) = 55

When $t = 55$, $P = 2500e^{0.0188 \times 55} = 7030$ million

Example

Radioactive strontium 90 is used in nuclear reactors and decays according to

$$A = Pe^{-0.0248t}$$

where P is the amount present at $t = 0$ and A is the amount left after t years. Find the time required for 10 kilograms to decay to 5 kilograms.

Solution

$$A = 10e^{-0.0248t}$$

$$5 = 10e^{-0.0248t}$$

$$0.5 = e^{-0.0248t}$$

$$\ln 0.5 = -0.0248t$$

$$t = \frac{\ln 0.5}{-0.0248}$$

$$= 27.95 \text{ years}$$

Example

The rate at which a chemical is generated in a reaction is given by

$$x = 11.6 \times 2.7^{1-2.4t}$$

where x is the amount of chemical present at time t.

A graph of x against t is very difficult to draw. Use logs to obtain a simpler graph.

Solution

$$x = 11.6 \times 2.7^{1-2.4t}$$

$$\log x = \log (11.6 \times 2.7^{1-2.4t})$$

$$= \log 11.6 + \log 2.7^{1-2.4t} = \log 11.6 + (1 - 2.4t) \log 2.7$$

$$= 1.064 + 0.4314(1 - 2.4t)$$

$$= 1.064 + 0.4314 - 0.4314 \times 2.4t$$

$$= 1.064 + 0.4314 - 1.0350t$$

$$= 1.496 - 1.035t$$

This function is a straight-line graph of $\log x$ against t.

Exercise 4.9

1. The acidity of a solution is measured by the symbol pH, where pH is –log of the concentration of the hydrogen ion, i.e. pH = – log $[H^+]$.

   Find:

   i. the pH if $[H^+]$ is

      a. 10^{-3} b. 1 c. 2×10^{-5}

   ii. $[H^+]$ if the pH is

      a. 5 b. 3.4 c. 0.49

2. The average height of females (y) in metres is given approximately in terms of age (x) by:

   $y = 0.61 + 0.796 \log (x + 1)$, for $x \leq 20$

   a. Find the average height of an 8 year old girl.

   b. At what age is the average height 1.5 metres?

3. Use logs to transform the equations below into straight lines.

   a. $y = 1.4^x$ b. $y = 6 \times 1.2^x$ c. $y = 3 \times 0.6^{-2x}$ d. $y = 2 \times 3^{3-2x}$

Solutions

1.

   i. a. 3 b. 0 c. 4.699

   ii. a. 10^{-5} b. $10^{-3.4} = 3.981 \times 10^{-4}$ c. $10^{-0.49} = 0.324$

2. a. 1.37 m b. 12.1 years

3.

   a. $\log y = x \log 1.4 = 0.146x$

   b. $\log y = \log 6 + x \log 1.2 = 0.778 + 0.0792x$

   c. $\log y = 0.477 + 0.444x$

   d. $\log y = \log 2 + (3 - 2x) \log 3 = 1.732 - 0.954x$

Module 4: Self assessment

Questions 4.1

1. Write the expression $2^8 = 256$ in its logarithmic form
2. Use the logarithm rules to express the following as a single logarithm: $\log ab^3 - \log ab$
3. Given that $s = \ln(5t + 3)$, express t in terms of s
4. Make t the subject of $3s = 5e^{2t+1}$
5. Simplify $e^{2 \ln x}$
6. Sketch the graph of $y = e^x$
7. Population of a town is increasing according to the formula $P = P_o e^{kt}$ where P is the population in t years, P_o is the initial population and k is the growth constant. If the population in 1987 was 1000 and in 1992 was 1400, when will the population double?

Questions 4.2

1. Express $\log_2 147 = 7.2$ in an exponential form
2. Use the logarithm rules to express the following as a single logarithm:
$\frac{1}{3}\log_2 27 + \frac{1}{2}\log_2 16 - \log_2 8$
3. Simplify $\ln e^{5x}$
4. Given that $y = \frac{1}{2}e^{2x+1}$, express x in terms of y.
5. Make t the subject of $2s = \frac{1}{2}\log_{10} t + 3$
6. Sketch the graph of $y = e^{-x}$
7. The temperature T at any time t of a steel ingot cooling from 350°C to 15°C is modelled by $T = 15 + 350e^{-kt}$. If after 3 hours the temperature is 150°C, what would the temperature be after 10 hours?

Module 5 – Trigonometry

Objectives

In this module you are required to be able to:

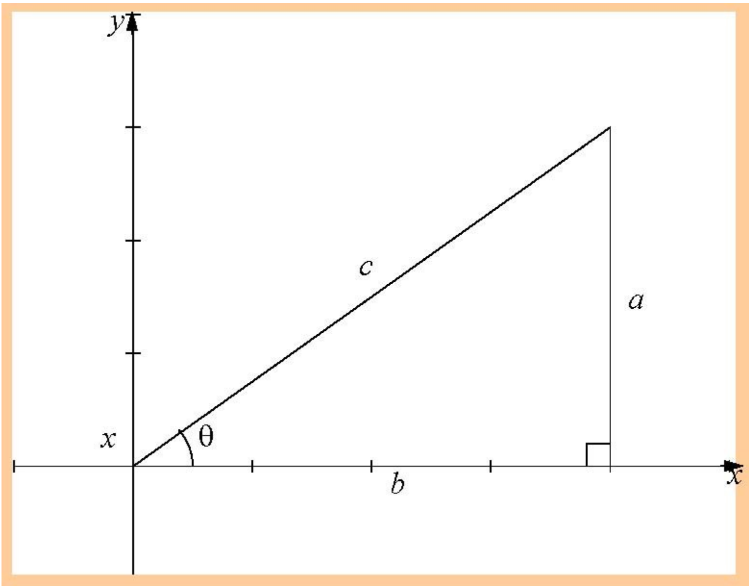
- evaluate trigonometric ratios for any right-angled triangle
- evaluate trigonometric ratios for positive or negative angles of any size
- evaluate positive or negative inverse trigonometric ratios
- solve problems involving trigonometric ratios and their relationships
- apply the sine rule and cosine rule to find sides and angles for any triangle
- find the area of any triangle
- understand the meaning of radians and convert angles from one system to the other
- find the area of the sector of a circle
- know the general shape of graphs or trigonometric functions
- draw a trigonometric graph by plotting selected points.

5.1 Trigonometric relationships

There are six trig. ratios: sine θ, cosine θ, tangent θ, cosecant θ, secant θ and cotangent θ.

$$\text{tangent}\,\theta = \frac{\text{sine}\,\theta}{\text{cosine}\,\theta}$$

These ratios are abbreviated as: sin θ, cos θ, tan θ, cosec θ, sec θ and cot θ. These symbols are referred to as ratios since they can be expressed in terms of the sides of a **right-angled triangle**.



$$\sin\theta = \frac{a}{c} \qquad \cos\theta = \frac{b}{c} \qquad \tan\theta = \frac{a}{b} = \frac{\sin\theta}{\cos\theta}$$

$$\text{and} \quad \csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

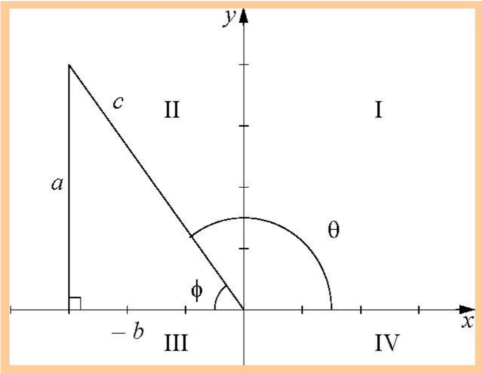
These trig. terms can be evaluated on any scientific calculator. Note that most machines work in ‘decimal’ degrees rather than degrees, minutes and seconds which must be converted to decimal fractions of degrees before calculation, e.g.

$$\sin 24^\circ 12'2'' = \sin\left(24 + \frac{12}{60} + \frac{2}{60\times 60}\right)^\circ = \sin 24.200556^\circ = 0.40993$$

Your calculator can accept positive or negative angles (a negative angle means that it is measured clockwise instead of anticlockwise) of any size. However, the trig. ratios of angles outside the range of 0° to 90° i.e. outside quadrant I, cannot be interpreted by the triangle diagram above. In fact many of the results are negative. These values may be found by considering the cases following.

a. **90° to 180°**

i.e. angles of quadrant II



Note: the Cartesian co-ordinate system divides the XY plane into 4 quadrants.

In the second quadrant *b* is negative and *a* is positive.

Thus, $\sin\theta = \frac{a}{c}$ $\cos\theta = -\frac{b}{c}$ $\tan\theta = \frac{a}{b}$

Note that in this quadrant,

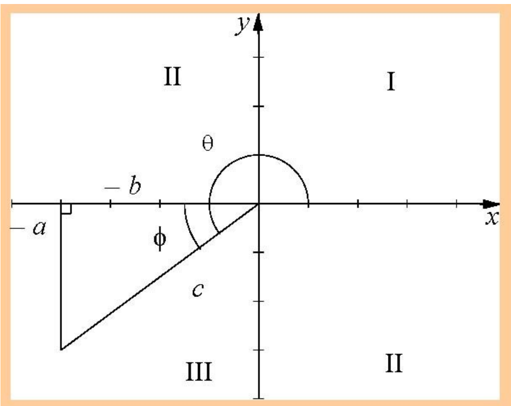
$\sin\Phi = \sin\theta$

Thus,

$\sin(180° - \theta) = \sin\theta$

b. **180° to 270°**

i.e. angles of quadrant III



In the third quadrant, both *a* and *b* are negative which means that sin θ and cos θ are both negative and tan θ is positive. Note that

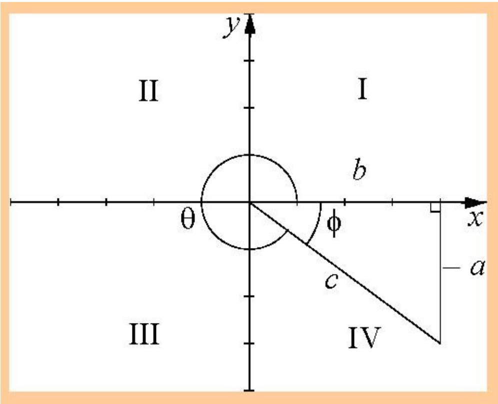
$$\tan\Phi = \frac{a}{b} = \tan\theta$$

Thus,

$\tan(180° + \theta) = \tan\theta$

c. **270° to 360°**

i.e. angles of quadrant IV



In this quadrant a is negative and b is positive thus $\cos\theta$ is positive and $\sin\theta$ and $\tan\theta$ are both negative.

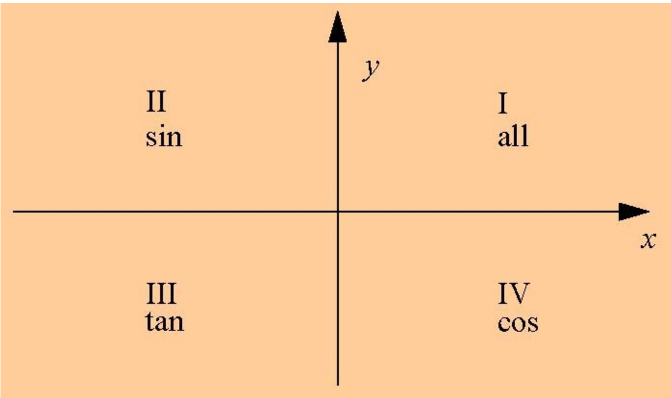
Note that

$\cos\theta = \cos\Phi$

Thus,

$\cos(360° - \theta) = \cos\theta$

These results may be summarised by the diagram below which gives the **positive** ratios in each quadrant.



Useful mnemonics to remember which ratios are positive are

All **S**tations **T**o **C**entral

All **S**tudents **T**ake **C**hemistry

This diagram also illustrates that

$\sin(-\theta) = -\sin\theta$,

$\cos(-\theta) = \cos\theta$, and

$\tan(-\theta) = -\tan\theta$.

Thus there are usually two angles between 0° and 360° which have the same trig ratio (except for $\sin\theta = \pm$ and $\cos\theta = \pm 1$). For example, if we wish to find θ such that $\sin\theta = 0.5$, then $\theta = 30°$ or 150°, since $\sin\theta = \sin(180° - \theta)$.

Given $\sin\theta$ we can find θ by using the INV key followed by the SIN key on the calculator. This gives us the SIN^{-1} key. The result is an angle between − 90° and 90°. From this first value of θ, we may find the other angle between 0° and 360° which also satisfies the trig equation.

For example if $\sin\theta = 0.5$, the calculator shows that $\theta = 30°$.

The other angle must be found using our knowledge of trig relationships. We know $\sin\theta = \sin(180° - \theta)$.

$\therefore$ If $\theta_1 = 30°$ gives $\sin\theta_1 = 0.5$, then $\theta_2 = (180° - 30°)$ also gives $\sin\theta_2 = 0.5$.

So the two angles which satisfy the trig equation $\sin\theta = 0.5$ are 30° and 150°.

Similarly to find two angles between 0° and 360° such that $\sin\theta = -0.5$, we find $\sin^{-1}(-0.5) = -30°$ from the calculator

$\therefore \sin(180° - (-30°)) = 0.5$ i.e. the other angle is 210°.

So the two angles which satisfy $\sin\theta = -0.5$ are − 30° and 210°.

To obtain angles between 0° and 360° we note that − 30° is the same as 330°.

$\therefore$ The two angles are 210° and 330°.

The angle given by calculators on pressing 0.5 INV SIN is the 'inverse sin of 0.5' written as '$\sin^{-1} 0.5$' or 'arcsin 0.5'.

Note that $\sin^{-1} 0.5$ **does not** mean $\dfrac{1}{\sin 0.5}$	$\sin^{-1} 0.5$ means the angle whose sin is 0.5

The other trig ratios also have inverses, i.e. arccos x, arctan x, arcsec x, arccosec x, arcot x where x is a number, may be defined similarly. These inverse trig ratios have a **unique value**.

This implies that an equation like $\sin\theta = 0.5$ has two solutions for θ in the range 0° to 360° (namely 30° and 150°) but arcsin 0.5 has only one value, 30°.

In order to find the second angle when solving trig equations, you must know the diagram showing the positive quadrants and the relationships:

$\sin(180 - \theta) = \sin\theta$

$\cos(360 - \theta) = \cos\theta$

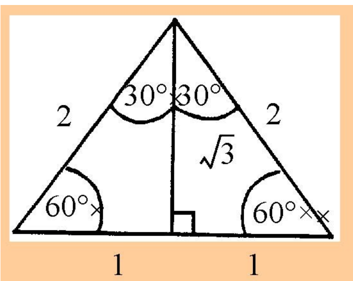
$\tan(180 + \theta) = \tan\theta$

The values of the trigonometric ratios for some angles commonly used are given in the table below.

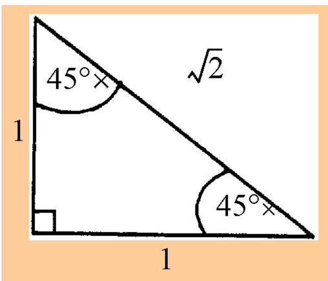
Sin θ, cos θ and tan θ for angles 0° to 360° given in the table should be memorised.

	θ							
	0°	**30°**	**45°**	**60°**	**90°**	**180°**	**270°**	**360°**
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	−1	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0	1
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined	0	Not defined	0
sec θ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined	−1	Not defined	1
csc θ	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	Not defined	−1	Not defined
cot θ	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	Not defined	0	Not defined

Two triangles which are used by many people to help them remember the trig ratios for angles of 30°, 60° and 45° are given below.



Equilateral triangle



Right Angled Isosceles triangle

Exercise 5.1

Evaluate

a. $\cos 72° 4'$ b. $\sec 105.37°$ c. $\tan(-15° 23' 32")$

d. $\arctan 0.3$ e. $\arccos 0.3$ f. $\tan^{-1} 2.7$

g. $\sin^{-1}(-0.6)$ h. $\cot^{-1} 2$

Solutions

Note: There are two systems for measuring angles: radians and degrees. Make sure that the calculator is switched to degrees.

a. $72° 4' = \left(72 + \dfrac{4}{60}\right)^{\circ} = 72.0667°$

$\cos 72.0667° = 0.3079$

b. $\sec 105.37^{\circ} = \dfrac{1}{\cos 105.37^{\circ}} = \dfrac{1}{-0.2651} = -3.7729$

c. $15^{\circ} 23' 32'' = \left(15 + \dfrac{23}{60} + \dfrac{32}{60^2}\right)^{\circ} = (15 + 0.383333 + 0.008889)^{\circ} = 15.392222^{\circ}$

$\tan(-15.392222) = -0.2753$

d. $\arctan 0.3 = 16.699°$

Note that $\arctan 0.3$ is the same as $\tan^{-1} 0.3$

e. $\arccos 0.3 = 72.54°$

f. $\tan^{-1} 2.7 = 69.68°$

Note that $\tan^{-1} x$ is not the same as $\dfrac{1}{\tan x}$

g. $\sin^{-1}(-0.6) = -36.87°$

360° may be added to this if a positive answer is preferred, i.e. $360 - 36.87 = 323.13°$.

The angle $-36.87°$ is the same angle as $323.13°$.

h. If $\cot\theta = 2$, then $\dfrac{1}{\cot\theta} = \tan\theta = \dfrac{1}{2} = 0.5$

$\therefore \cot^{-1} 2$ is the same angle as $\tan^{-1} 0.5 = 26.57°$

Exercise 5.2

Find all angles in a 360° circle which satisfy the trig equations:

a. $\sin\theta = 0.3$	b. $\tan\theta = 1.3$	c. $\cos\theta = 0.8$
d. $\sin\theta = -0.6$	e. $\tan\theta = -2.7$	f. $\cos\theta = -0.2$

Solutions

a. One solution is $\theta = \arcsin 0.3 = 17.46°$ and the other is $(180° - \theta) = 180° - 17.46° = 162.54°$

[Check on your calculator that sin of both these angles is 0.3]

b. Solutions are $\theta = \arctan 1.3$ and $(180° + \theta)$

$= 52.43°$ and $(180° + 52.43°)$

$= 52.43°$ and $232.43°$

[Check on calculator]

c. $\theta = \arccos 0.8$ and $(360° - \theta)$

$= 36.87°$ and $(360° - 36.87°)$

$= 36.87°$ and $323.13°$

[Check on calculator]

d. $\theta = \arcsin(-0.6)$ and $(180° - \theta)$

$= -36.87°$ and $(180° - (-36.87°))$

$= -36.87°$ and $216.37°$

[Note that $-36.87°$ may also be written as $360° - 36.87° = 323.13°$]

e. $\theta = \arctan(-2.7)$ and $(180° + \theta)$

$= -69.68°$ and $(180° - 69.68°)$

$= -69.68°$ and $110.32°$

[Note that $-69.68°$ may also be written as $360° - 69.68° = 290.32°$]

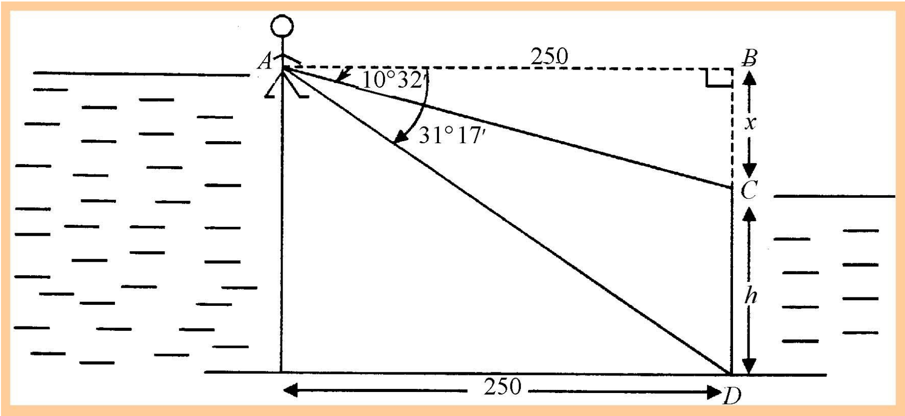
f. $\theta = \arccos(-0.2)$ and $(360° - \theta)$

$= 101.54°$ and $(360° - 101.54)°$

$= 101.54°$ and $258.46°$

Exercise 5.3

A surveyor on the roof of a building observes that the top of another building below forms an angle of 10° 32' with the horizontal whereas the base of the building forms an angle of 31° 17' to the horizontal. If the buildings are 250 metres apart, find the height of the observed building.



Solution

From the triangle ABC,

$$\tan 10° \, 32' = \frac{x}{250}$$

$$\therefore x = 250 \tan 10° \, 32' = 250 \tan 10.53333° = 250 \times 0.1859 = 46.49 \text{ m}$$

From the triangle ABD,

$$x + h = 250 \tan 31° \, 17'$$

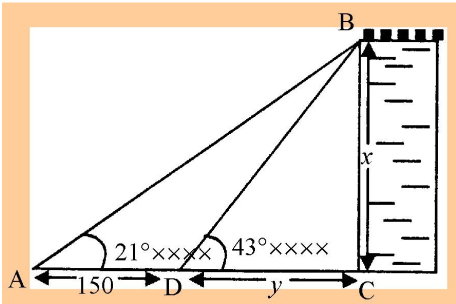
$$\therefore 46.49 + h = 250 \tan 31.283333° = 250 \times 0.6076 = 151.90 \text{ m}$$

$$\therefore h = 151.90 - 46.49 = 105.41 \text{ m}$$

Exercise 5.4

The angle to the top of a tower is 21° above horizontal and from another point 150 m closer it is 43°. Find the height of the tower.

Solution



Let x be the height of the tower and y be the distance from the second point to the tower.

From the triangle DBC,

$$\tan 43^\circ = \frac{x}{y} \therefore y = \frac{x}{\tan 43^\circ} = \frac{x}{0.9325}$$

From the triangle ABC

$$\tan 21^\circ = \frac{x}{150 + y} \therefore \tan 21^\circ (150 + y) = x$$

$$\text{i.e. } \tan 21^\circ \left(150 + \frac{x}{0.9325}\right) = x$$

$$\therefore 0.3839\left(150 + \frac{x}{0.9325}\right) = x$$

$$\therefore 0.3839 \times 150 + \frac{0.3839x}{0.9325} = x$$

$$\therefore 57.585 + 0.4117x = x$$

$$\therefore 57.585 = x(1 - 0.4117)$$

$$= 0.5883x$$

$$\therefore x = \frac{57.585}{0.5883} = 97.9 \text{ m}$$

Exercise 5.5

Simplify

a. cos A. tan A. cosec A. b. $\dfrac{\sec A}{\operatorname{cosec} A}$

Solutions

a. $\cos A \tan A \operatorname{cosec} A = \cos A \dfrac{\sin A}{\cos A} \dfrac{1}{\sin A} = 1$

b. $\dfrac{\sec A}{\operatorname{cosec} A} = \dfrac{1/\cos A}{1/\sin A} = \dfrac{\sin A}{\cos A} = \tan A$

Additional exercises

1. Evaluate:

a. $\sin(-22° \, 15')$	b. $\cos 36.42°$	c. $\tan(-120° \, 36')$
d. $\sin 223.4°$	e. $\tan 96.7°$	f. $\cos 192° \, 33'22''$

2. Find the following angles:

a. $\arcsin 0.4$	b. $\cos^{-1} 0.2$	c. $\tan^{-1} 1.9$
d. $\arcsin(-0.25)$	e. $\arccos(-0.51)$	f. $\arctan(-2.6)$

3. Find all angles between 0° and 360° which satisfy the following equations:

a. $\tan\theta = 1.3$	b. $\sin\theta = 0.63$	c. $\cos\theta = 0.13$
d. $\sin\theta = -0.28$	e. $\tan\theta = -1.6$	f. $\cos\theta = -0.84$

4. From one point on the ground the angle to the top of a tower is 32° above the horizontal. From a point 200 m closer, the angle is 64°. Find the height of the tower.

Solutions

1.

a. -0.3786	b. 0.8047	c. 1.6909
d. -0.6871	e. -8.5126	f. -0.9761

2.

a. 23.578°	b. 78.46°	c. 62.24°
d. $-14.48°$	e. 120.66°	f. $-68.96°$

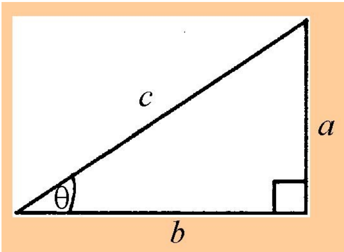
3.

a. 52.43° and 232.43°	b. 39.05° and 140.95°
c. 82.53° and 277.47°	d. 343.74° and 196.26°
e. 302.01° and 122.01°	f. 147.14° and 212.86°

4. 179.76 m

5.2 Trigonometric identities

Some important relationships between the trig ratios can be established using Pythagoras' Theorem.



$$\sin\theta = \frac{a}{c}, \quad \cos\theta = \frac{b}{c}$$

$$\therefore \sin^2\theta + \cos^2\theta = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1$$

$$\sin^2\theta + \cos^2\theta = 1$$

Note: $\sin^2\theta$ means $\sin\theta$ multiplied by $\sin\theta$, i.e. $\sin^2\theta = (\sin\theta)^2$. It does **not** mean $\sin(\theta^2)$.
$\sin^2\theta + \cos^2\theta = 1$ may be rewritten as $\sin^2\theta = 1 - \cos^2\theta$, or
$\cos^2\theta = 1 - \sin^2\theta$

Dividing by $\cos^2\theta$ gives

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}$$

$$\therefore \boxed{\tan^2\theta + 1 = \sec^2\theta}$$

Dividing by $\sin^2\theta$ gives

$$\boxed{\cot^2\theta + 1 = \operatorname{cosec}^2\theta}$$

Exercise 5.6

Simplify

a. $\dfrac{1}{\cos^2\theta} - \tan^2\theta$

b. $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$

c. $\dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$

Solutions

a. $\dfrac{1}{\cos^2\theta} - \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{1-\sin^2\theta}{\cos^2\theta} = \dfrac{\cos^2\theta}{\cos^2\theta} = 1$

b. $\sin^2 x + \cos^2 x + 2\sin x\cos x + \sin^2 x + \cos^2 x - 2\sin x\cos x = 2\sin^2 x + 2\cos^2 x$

$= 2(\sin^2 x + \cos^2 x) = 2 \times 1 = 2$

c. $\dfrac{\tan\theta}{\sqrt{\sec^2\theta}} = \dfrac{\tan\theta}{\sec\theta} = \dfrac{\sin\theta/\cos\theta}{1/\cos\theta} = \sin\theta$

Exercise 5.7

Find all angles between 0° and 360° which satisfy

a. $2\cos^2\theta - \sin\theta - 1 = 0$

b. $\sec^2\theta = 3 + \tan\theta$

Solutions

a.
$$\begin{aligned}
2(1-\sin^2\theta) - \sin\theta - 1 &= 0\\
2 - 2\sin^2\theta - \sin\theta - 1 &= 0\\
-2\sin^2\theta - \sin\theta + 1 &= 0\\
2\sin^2\theta + \sin\theta - 1 &= 0\\
(2\sin\theta - 1)(\sin\theta + 1) &= 0
\end{aligned}$$

$\therefore \sin\theta = \dfrac{1}{2}$ or $\sin\theta = -1$

$\sin\theta = \dfrac{1}{2} \Rightarrow \theta = \sin^{-1} 0.5 \quad \therefore \theta = 30°$ and $(180° - \theta) = \sin^{-1} 0.5$

$\therefore 150° = \sin^{-1} 0.5$

$\sin\theta = -1 \Rightarrow \theta = \sin^{-1}(-1) \therefore \theta = -90°$

and $-90°$ is the same as $360° - 90° = 270°$

$\therefore$ Solutions are $\theta = 30°$, $150°$ and $270°$.

b.

$$\begin{aligned}
\sec^2\theta &= 3 + \tan\theta \\
1 + \tan^2\theta &= 3 + \tan\theta \\
1 + \tan^2\theta - 3 - \tan\theta &= 0 \\
\tan^2\theta - \tan\theta - 2 &= 0 \\
(\tan\theta - 2)(\tan\theta + 1) &= 0 \\
\therefore \tan\theta = 2 \text{ or } \tan\theta &= -1
\end{aligned}$$

$$\begin{aligned}
\tan\theta = 2 \Rightarrow \theta = \tan^{-1} 2 \therefore \theta = 63.43° \text{ and } \therefore (180° + \theta) &= \tan^{-1} 2 \\
\therefore (180° + 63.43°) &= \tan^{-1} 2 \\
\therefore 243.43° &= \tan^{-1} 2
\end{aligned}$$

$$\begin{aligned}
\text{from } \tan\theta = -1 \Rightarrow \theta &= \arctan(-1) \text{ or } (180° + \theta) \\
&= -45° \text{ or } (180 - 45°) \\
&= -45° \text{ or } 135°
\end{aligned}$$

and $-45°$ is the same as $360° - 45° = 315°$

$\therefore$ Solutions are $\theta = 63.43°, 135°, 243.43°$ and $315°$

Additional exercises

1. If $\tan A = 2$, find $\cos A$ using the trig identities.
2. If $\sin B = 0.4$, find $\cos B$ and $\cot B$ using the trig identities.
3. Prove that $\tan\theta + \cot\theta = \dfrac{1}{\sin\theta\cos\theta}$
4. Find all angles between 0° and 360° which satisfy
   a. $\cos^2\theta + 4\sin^2\theta = 2$
   b. $2\cos^2\theta = 3\sin\theta$
   c. $2\sin^2\theta - 9\cos\theta + 3 = 0$

Solutions

1. $\sec^2 A = 1 + \tan^2 A = 5$

   $\sec A = \pm\sqrt{5}$

   $\cos A = \dfrac{1}{\sec A} = \pm\dfrac{1}{\sqrt{5}} = \pm 0.4472$

2. $\cos^2 B = 1 - \sin^2 B = 1 - 0.4^2 = 0.84$

   $\therefore \cos B = \pm 0.9165$

   $\cot B = \dfrac{\cos B}{\sin B} = \pm\dfrac{0.9165}{0.4} = \pm 2.2913$

3. $\tan\theta + \cot\theta = \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\cos\theta\,\sin\theta} = \dfrac{1}{\sin\theta\,\cos\theta}$

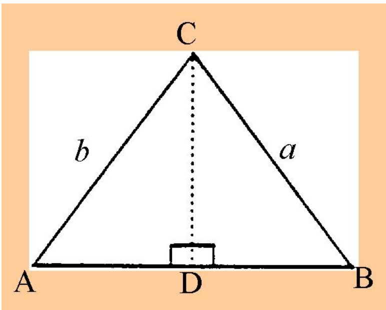
4. a. $\sin\theta = \pm\,0.5774$ $\quad\therefore \theta = 35.26°, 144.74°, 215.26°$ or $324.73°$

   b. $\sin\theta = 0.5$ or -2 Now $\sin\theta = -2$ is impossible $\therefore \theta = 30°$ or $150°$

   c. $\cos\theta = 0.5$ or -5 Now $\cos\theta = -5$ is impossible $\therefore \theta = 60°$ or $300°$

5.3 Triangle solution

One of the most common applications of trigonometry is to the problem of triangle ‘solution’. In this problem, some sides and angles of a triangle (not necessarily right-angled) are known and we want to find the remaining sides and angles.



In the triangle ACD, CD = $b \sin A$

In the triangle BCD, CD = $a \sin B$

$$\therefore b \sin A = a \sin B$$

$$\therefore \frac{b}{\sin B} = \frac{a}{\sin A}$$

Similarly, $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

These can be combined in the **sine rule**

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$

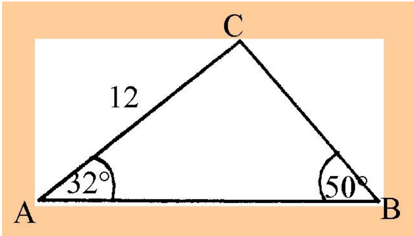
where a, b, c are the sides and A, B, C are the opposite angles of **any** triangle.

To use the sine rule, we need any 2 sides and an **opposite** angle or any 2 angles and an **opposite** side.

Exercise 5.8

Find all angles and sides of the triangles:

a.



b. AC = 11, BC = 9, ∠A = 50°

Solutions

a.

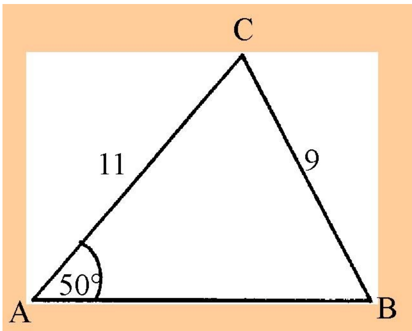
$$\frac{a}{\sin 32^\circ} = \frac{12}{\sin 50^\circ}$$

$$\therefore\ a = \frac{12\sin 32^\circ}{\sin 50^\circ} = \frac{12 \times 0.5299}{0.7660} = 8.301$$

$$\angle C = 180 - 32 - 50 = 98^\circ,\quad \frac{c}{\sin 98^\circ} = \frac{12}{\sin 50^\circ}$$

$$c = \frac{12\sin 98^\circ}{\sin 50^\circ} = \frac{12 \times 0.9903}{0.7660} = 15.512$$

b.



$$\frac{11}{\sin B} = \frac{9}{\sin 50^\circ} \quad \therefore \frac{\sin B}{11} = \frac{\sin 50^\circ}{9}$$

$$\therefore\ \sin B = \frac{11\sin 50^\circ}{9}$$

$$= \frac{11 \times 0.7660}{9} = 0.9363$$

$\therefore$ B = $\sin^{-1}$ 0.9363 or (180° – B)

= 69.435° or (180° – 69.435°)

= 69.435° or 110.565°

Note that both angles are possible so the triangle above is only one representation of the given information.

If B = 69.435°, then

$$\angle C = 180 - 50 - 69.435 = 60.565^\circ$$

$$\frac{c}{\sin 60.565^\circ} = \frac{9}{\sin 50^\circ}$$

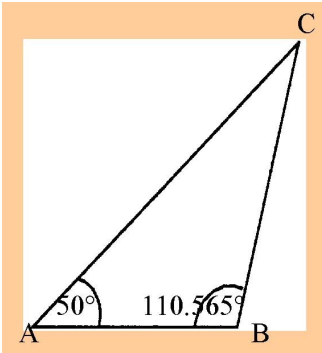
$$\therefore\ c = \frac{9\sin 60.565^\circ}{\sin 50^\circ} = \frac{9 \times 0.8709}{0.7660} = 10.232$$

If B = 110.565°, then

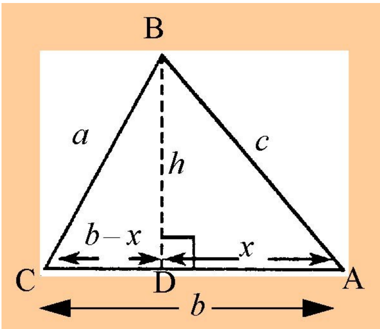
$$\angle C = 180 - 50 - 110.565 = 19.435^\circ$$

$$\frac{c}{\sin 19.435^\circ} = \frac{9}{\sin 50^\circ}$$

$$\therefore\ c = \frac{9\sin 19.435^\circ}{\sin 50^\circ} = 3.909$$



Cosine rule



In the triangle BCD, $a^2 = (b - x)^2 + h^2$

$$= b^2 - 2bx + x^2 + h^2$$

$$= b^2 - 2bx + c^2 \qquad \{\text{In triangle ADB, } x^2 + h^2 = c^2\}$$

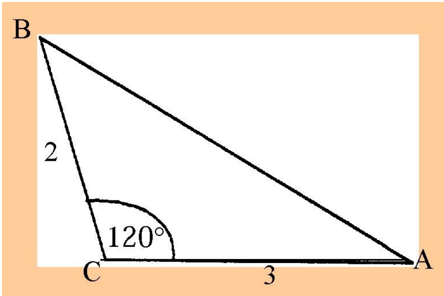
But $x = c$ cos A. Substituting gives the **cosine rule**

$$\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$$

To use the cosine rule, we require any 2 sides and the **included** angle or all 3 sides.

Exercise 5.9

Find all sides and angles of the triangle



Solution

Note that the only angle given is labelled C in the diagram which does not agree with the labelling in cosine rule. We may either re-label the diagram to make this A or adjust the letters in cosine rule to give:

$$c^2 = a^2 + b^2 - 2ab\cos C = 2^2 + 3^2 - 2 \times 2 \times 3\cos 120°$$

$$= 4 + 9 - 12 \times (-0.5)$$

$$= 13 + 6 = 19$$

$\therefore c = \sqrt{19} = 4.359$

[Note: only the positive square root is feasible for the length of a side of a triangle.]

$$\frac{4.359}{\sin 120^\circ} = \frac{3}{\sin B} \quad \therefore \ \sin B = \frac{3\sin 120^\circ}{4.359} = \frac{3 \times 0.8660}{4.359} = 0.596$$

$\therefore$ B = $\sin^{-1} 0.596$ or (180° – B) = 36.585° or (180° – 36.585°)

= 36.585° or 143.415°

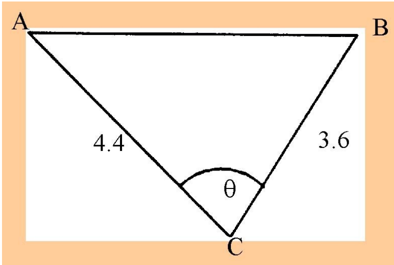
From the diagram, 143.415° is clearly impossible. Thus B = 36.585°.

A = 180° – 120° – 36.585° = 23.415°.

Exercise 5.10

A weight hangs from the junction of two ropes 4.4 metres and 3.6 metres long respectively. If the other ends of the ropes are attached to a beam at points 5 metres apart, find the angle between the ropes.

Solution



$$5^2 = 4.4^2 + 3.6^2 - 2 \times 4.4 \times 3.6 \cos\theta$$

$$25 = 19.36 + 12.96 - 31.68 \cos\theta$$

$$25 = 32.32 - 31.68 \cos\theta$$

$$25 - 32.32 = -31.68 \cos\theta$$

$$-7.32 = -31.68 \cos\theta$$

$$\cos\theta = \frac{7.32}{31.68} = 0.2311$$

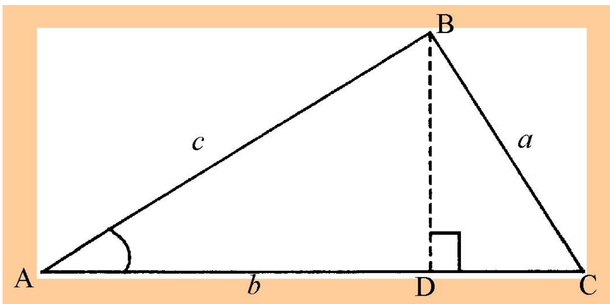
$$\therefore \theta = \cos^{-1}(0.2311) \text{ or } (360° - \theta)$$

$$= 76.64° \text{ or } 283.36°$$

Obviously, 283.36° is impossible. Thus, the angle is 76.64°.

Triangle area

The area of any triangle ABC, may be found as follows:



$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} b \times \text{BD}$$

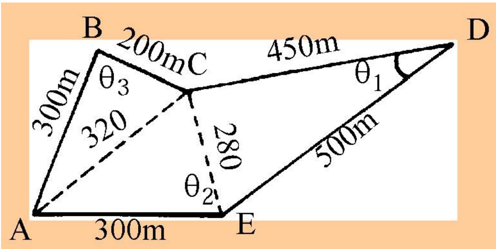
$$\text{But } \frac{\text{BD}}{c} = \sin A$$

$$\text{Area} = \frac{1}{2}bc\sin A$$

Thus, to find the area we require any 2 sides and the **included** angle.

Exercise 5.11

A geological exploration company takes out a mining lease of the shape below. Find its area.



Solution

In triangle CDE,

$$280^2 = 450^2 + 500^2 - 2 \times 450 \times 500 \cos \theta_1$$

$$78400 = 202500 + 250000 - 450000 \cos \theta_1$$

$$78400 = 452500 - 450000 \cos \theta_1$$

$$78400 - 452500 = -450000 \cos \theta_1$$

$$-374100 = -450000 \cos \theta_1$$

$$\therefore \cos \theta_1 = \frac{374100}{450000} = 0.8313$$

$$\therefore \theta_1 = \cos^{-1} 0.8313 \text{ or } (360° - \theta_1)$$

$$= 33.764° \text{ or } 326.24°$$

From the diagram, θ_1 must be 33.764°

$$\text{Area of triangle CDE} = \frac{1}{2} \times 450 \times 500 \sin 33.764°$$

$$= 112500 \times 0.5558 = 62525 \text{ m}^2$$

In triangle CAE,

$$320^2 = 280^2 + 300^2 - 2 \times 280 \times 300 \times \cos \theta_2$$

$$102400 = 78400 + 90000 - 168000 \cos \theta_2$$

$$-66000 = -168000 \cos \theta_2$$

$$\cos\theta_2 = 0.3929$$

$$\theta_2 = 66.868° \qquad \text{(The other answer of 293.13° is impossible)}$$

$$\text{Area} = \frac{1}{2} \times 300 \times 280 \sin 66.868°$$

$$= 42000 \times 0.9196 = 38623 \text{ m}^2$$

In triangle BCA,

$$320^2 = 300^2 + 200^2 - 2 \times 300 \times 200 \cos\theta_3$$

$$102400 = 90000 + 40000 - 120000 \cos\theta_3$$

$$-27600 = -120000 \cos\theta_3$$

$$\cos\theta_3 = 0.23$$

$$\theta_3 = 76.703°$$

$$\text{Area} = \frac{1}{2} \times 300 \times 200 \sin 76.703° = 29196 \text{ m}^2$$

$$\therefore \text{Total area of lease} = 62525 + 38623 + 29196 = 130344 \text{ m}^2$$

Additional exercises

1. Find the remaining angles of the triangle ABC given that:

   a. $b = 4, c = 2$, A = 30° b. $b = 5, c = 4$, A = 42°

   c. $a = 4, b = 7$, C = 110°

2. Find the remaining side of the triangle ABC given that:

   a. $c = 56, a = 74$, C = 35° 15′ b. $b = 15, c = 14$, B = 57° 22′

   c. $b = 6, c = 5$, B = 53° 8′

3. Find the angle A in the triangle ABC with the sides:

   a. $a = 2, b = 4, c = 3$ b. $a = 9, b = 5, c = 6$

4. Find all the remaining angles and sides of the triangle ABC where:

   a. $a = 2, b = 4$, C = 30° b. $a = 6, c = 7$, B = 75° 31′

5. Find the areas of the triangles in question 1.
6. Find the area of the quadrilateral ABCD given that:
   AB = 12, BC = 10, CD = 14, DA = 8, AC = 16.

Solutions

1.
   a. B = 126.21°, C = 23.79°
   b. B = 85.14°, C = 52.86°
   c. A = 24.19°, B = 45.81°

2.
   a. b = 96.65 or 24.21
   b. a = 16.82
   c. a = 7.47

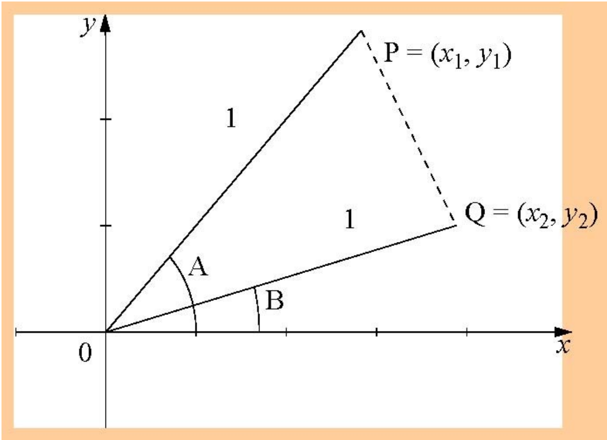
3.
   a. A = 28.96°
   b. 109.47°

4.
   a. A = 23.79°, B = 126.21°, c = 2.48
   b. A = 46.57°, C = 57.91°, b = 8.00

5. a. 2 b. 6.691 c. 13.16

6. 115.92

5.4 Compound angles



$\cos A = \frac{x_1}{1} = x_1$

$\sin A = \frac{y_1}{1} = y_1$

$\cos B = \frac{x_2}{1} = x_2$

$\cos B = \frac{y_2}{1} = y_2$

The coordinates of P are (cos A, sin A) and of Q are (cos B, sin B).
The distance PQ is given by

$$PQ^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$$
$$= (\cos^2 A + \cos^2 B - 2\cos A\cos B) + (\sin^2 A + \sin^2 B - 2\sin A\sin B)$$
$$= 1 - 2\cos A\cos B + 1 - 2\sin A\sin B$$
$$= 2 - 2\cos A\cos B - 2\sin A\sin B$$

From the cosine rule in triangle POQ,

$$PQ^2 = 1^2 + 1^2 - 2\times 1\times 1\cos(A - B) = 2 - 2\cos(A - B)$$
$$\therefore 2 - 2\cos A\cos B - 2\sin A\sin B = 2 - 2\cos(A - B)$$

$$\boxed{\cos(A - B) = \cos A\cos B + \sin A\sin B}$$

If we write (– B) for B we get

$$\boxed{\cos(A + B) = \cos A\cos B - \sin A\sin B}$$

We can derive a similar rule for the sin of a compound angle as follows:

$$\sin(A + B) = \cos(90° - (A + B)) = \cos((90° - A) - B)$$
$$= \cos(90° - A)\cos B + \sin(90° - A)\sin B$$

Since cos (90° – A) = sin A and sin (90° – A) = cos A, we get

$$\boxed{\sin(A + B) = \sin A\cos B + \cos A\sin B}$$

and similarly

$$\boxed{\sin(A - B) = \sin A \cos B - \cos A \sin B}$$

If we put B = A, we obtain the special results for double angles.

sin (A + A) = sin A cos A + cos A sin A

$$\boxed{\sin 2A = 2 \sin A \cos A}$$

cos (A + A) = cos A cos A – sin A sin A

$$\boxed{\cos 2A = \cos^2 A - \sin^2 A}$$

A summary of the results discussed to this point is given below:

A. Definitions and basic identities

$$\sin\theta = \frac{y}{r} = \frac{1}{\operatorname{cosec}\theta}$$

$$\cos\theta = \frac{x}{r} = \frac{1}{\sec\theta}$$

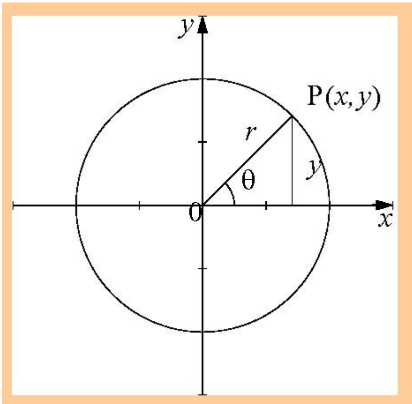
$$\tan\theta = \frac{y}{x} = \frac{1}{\cot\theta}$$

$$\sin(-\theta) = -\sin\theta;\ \cos(-\theta) = \cos\theta$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta = 1 + \tan^2\theta$$

$$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$$



$$\sin 2\theta = 2\sin\theta\cos\theta;\ \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2};\ \sin^2\theta = \frac{1-\cos 2\theta}{2};$$

{Prove these by substituting for cos2θ}

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A - 90^\circ) = -\cos A;\ \cos(A - 90^\circ) = \sin A$$

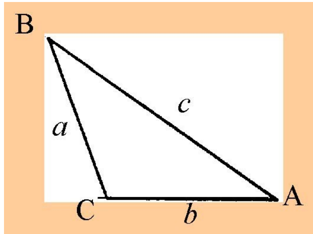
$$\sin(A + 90^\circ) = \cos A;\ \cos(A + 90^\circ) = -\sin A$$

B. Angles and sides of a triangle

Cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$

Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Area: $= \dfrac{1}{2}bc\sin A = \dfrac{1}{2}ac\sin B = \dfrac{1}{2}ab\sin C$



Exercise 5.12

Simplify

a. $\dfrac{\sin(A+B)}{\cos A\cos B}$

b. $\sqrt{\dfrac{1+\cos 2\theta}{1-\cos 2\theta}}$

c. $\dfrac{\sin 4\theta - \cos 2\theta}{1-\cos 4\theta - \sin 2\theta}$

Solutions

a.

$$\begin{aligned}\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B} &= \frac{\sin A\cos B}{\cos A\cos B}+\frac{\cos A\sin B}{\cos A\cos B}\\ &= \frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}\\ &= \tan A+\tan B\end{aligned}$$

b.

$$\begin{aligned}\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}} &= \sqrt{\frac{1+\cos^2\theta-\sin^2\theta}{1-\left(\cos^2\theta-\sin^2\theta\right)}}\\ &= \sqrt{\frac{1-\sin^2\theta+\cos^2\theta}{1-\cos^2\theta+\sin^2\theta}}\\ &= \sqrt{\frac{\cos^2\theta+\cos^2\theta}{\sin^2\theta+\sin^2\theta}}\\ &= \sqrt{\frac{2\cos^2\theta}{2\sin^2\theta}}\\ &= \sqrt{\frac{\cos^2\theta}{\sin^2\theta}}\\ &= \frac{\cos\theta}{\sin\theta}\\ &= \cot\theta\end{aligned}$$

c.

$$\frac{\sin 4\theta - \cos 2\theta}{1-\cos 4\theta - \sin 2\theta} = \frac{2\sin 2\theta \cos 2\theta - \cos 2\theta}{1-\left(\cos^2 2\theta - \sin^2 2\theta\right) - \sin 2\theta}$$

$$= \frac{\cos 2\theta\left(2\sin 2\theta - 1\right)}{1-\cos^2 2\theta + \sin^2 2\theta - \sin 2\theta}$$

$$= \frac{\cos 2\theta\left(2\sin 2\theta - 1\right)}{\sin^2 2\theta + \sin^2 2\theta - \sin 2\theta}$$

$$= \frac{\cos 2\theta\left(2\sin 2\theta - 1\right)}{2\sin^2 2\theta - \sin 2\theta}$$

$$= \frac{\cos 2\theta\left(2\sin 2\theta - 1\right)}{\sin 2\theta\left(2\sin 2\theta - 1\right)}$$

$$= \frac{\cos 2\theta}{\sin 2\theta} = \cot 2\theta$$

Additional exercises

1. Write the following in terms of sin A, cos A, sin B and cos B.

   a. cos (A + 2B) b. sin (2A + B) c. cos (2B – A)

2. Express the following as a single trig ratio.

   a. cos 24° sin 35° – sin 24° cos 35°

   b. $\cos^2 45° - \sin^2 45°$

   c. cos 32° cos 58° – sin 32° sin 58°

   d. sin 21° cos 46° + cos 21° sin 46°

3. Prove the identities:

   a. $\cot A + \cot B = \dfrac{\sin(A+B)}{\sin A \sin B}$

   b. $\dfrac{\sin(A-B)}{\cos(A+B)} = \dfrac{\tan A - \tan B}{1-\tan A \tan B}$

Solutions

1. a. $\cos A \cos^2 B - \cos A \sin^2 B - 2 \sin A \sin B \cos B$

   b. $2 \sin A \cos A \cos B + \cos^2 A \sin B - \sin^2 A \sin B$

   c. $2 \sin A \sin B \cos B + \cos A \cos^2 B - \cos A \sin^2 B$

2.

   a. sin (35° – 24°) = sin 11° b. cos (2 × 45°) = cos 90°

   c. cos (32° + 58°) = cos 90° d. sin (21° + 46°) = sin 67°

3. a.

$$\cot A + \cot B = \frac{\cos A}{\sin A} + \frac{\cos B}{\sin B} = \frac{\cos A \sin B + \sin A \cos B}{\sin A \sin B} = \frac{\sin(A+B)}{\sin A \sin B}$$

b.

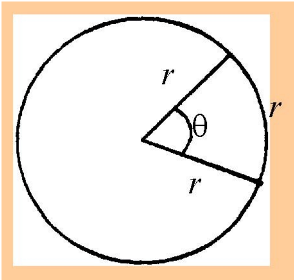
$$\frac{\sin(A-B)}{\cos(A+B)} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{\dfrac{\sin A \cos B}{\cos A \cos B} - \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}} \quad \text{(Divide each term by } \cos A \cos B\text{)}$$

$$= \frac{\tan A - \tan B}{1 - \tan A \tan B}$$

5.5 Radian measure

Radians and degrees are alternative means of measuring the size of an angle.



Consider a circle of radius r. The angle θ will be one radian if it corresponds to an arc of the circumference of length r. This means that

1 radian = 57.2058 degrees i.e. ($1^c \equiv 57.2958°$)

Note that a radian is really a ratio of arclength to radius although it corresponds to a measure of angle. Thus it may be mixed with other measurements in an equation whereas degree measurements cannot. Often the little c superscript to denote radians is omitted. If an angle does not have the degree sign you must assume it is in radians.

Radians may be converted to degrees and vice versa by use of the relationship

$$\boxed{\pi \text{ radians} = 180 \text{ degrees}}$$

For instance

$$60^\circ = 60 \times \frac{\pi}{180} = \frac{\pi}{3} = \frac{3.141593}{2} = 1.047198^c$$

It is not necessary to convert radians to degrees before using a calculator since most machines work directly in either system. Most scientific and engineering problems are expressed in radians.

Make sure your calculator is in the correct mode. In future work we will be using radians.

Exercise 5.13

Convert

a. 217° to radians

b. 2.64^c to degrees.

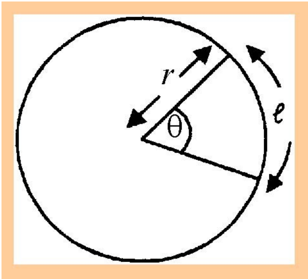
Solutions

a. $217 \times \dfrac{\pi}{180} = 3.787^c$

b. $2.64 \times \dfrac{180}{\pi} = 151.26^\circ$

Length of arc of a circle

The length of the arc of a circle may be deduced as follows.



Let θ be the angle subtended at the centre of a circle of radius *r* by an arc of length *l*.

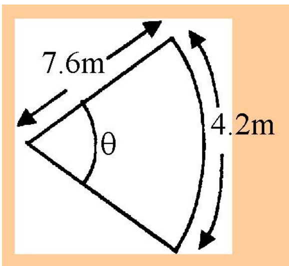
Then $\theta = \dfrac{l}{r}$ radians (by definition of a radian)

$\boxed{\therefore l = r\theta}$ where θ is measured in radians

Exercise 5.14

A circular arc of radius 7.6 metres has a length 4.2 metres. Find the angle subtended at the centre.

Solution



$$4.2 = 7.6\,\theta$$

$$\therefore \theta = \frac{4.2}{7.6} = 0.553^c$$

Area of a sector of a circle

We can derive a formula for the area of a sector of a circle as follows:

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi}$$

$$\therefore\ \text{area of sector} = \frac{\theta}{2\pi} \cdot \text{area of circle}$$

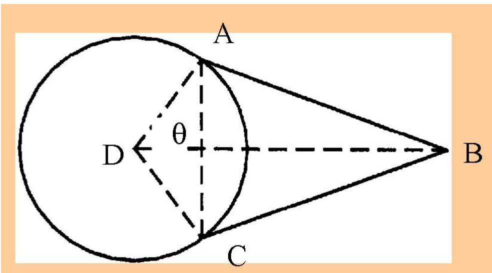
$$= \frac{\theta}{2\pi} \cdot \pi r^2$$

$$\boxed{\text{area of sector} = \frac{1}{2} r^2 \theta}$$

[Note that this formula **cannot** be used if θ is measured in degrees.]

Exercise 5.15

Find the area of the solid line figure with the dimensions given below:



AC = 3 m, AD = 2.5 m, AB = 2.4 m, CB = 2.4 m

Solution

In triangle DAC, $\sin\theta = \dfrac{1.5}{2.5} = 0.6 \quad \therefore \theta = \sin^{-1} 0.6 = 0.6435^c$

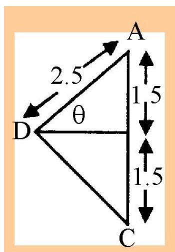
$$\therefore \angle ADC = 2\,\theta = 1.287^c$$

Outer part of circle from A to C has an angle at the centre of $(2\pi - 1.287)$ radians.

$$\text{Area of sector} = \frac{1}{2}\, r^2\, (2\pi - 1.287)$$

$$= \frac{2.5^2}{2} \times (2\pi - 1.287)$$

$$= 3.125 \times 4.996 = 15.613 \text{ m}^2$$



$$\text{Area of triangle DAC} = \frac{1}{2} \times 2.5^2 \sin 1.287^c = 3.125 \times 0.9600 = 3 \text{ m}^2$$

$$\text{In triangle ABC, } \sin \Phi = \frac{1.5}{2.4} = 0.625 \quad \therefore \Phi = 0.6751^c$$

$$\therefore \text{ Area of triangle ABC} = \frac{1}{2} \times 2.4^2 \times \sin (2 \times 0.6751^c)$$

$$= 2.88 \sin 1.3502^c$$

$$= 2.810 \text{ m}^2$$

$$\therefore \text{Total area} = 15.613 + 3 + 2.81 = 21.423 \text{ m}^2$$

Additional exercises

1. Express the following angles in radians

   a. 360° b. 30° c. 10° d. 137°15'12"

2. Express the following angles in degrees.

   a. $\frac{\pi}{3}^c$ b. 1.674^c c. $\frac{5\pi}{4}^c$ d. 6.803^c

3. Find the arclength and area of the sector of the circle with radius and angle given below:

   a. $r = 4$ m, $\theta = \frac{\pi}{3}$ b. $r = 5$ m, $\theta = 2.5$

   d. $r = 10$ m, $\theta = 45°$ c. $r = 6$ m, $\theta = 130°$

Solutions

1.

   a. $2\pi = 6.283^c$ b. $\frac{\pi}{6} = 0.524^c$ c. $\frac{\pi}{18} = 0.1745^c$ d. 2.3955^c

2.

   a. 60° b. 95.913° c. 225° d. 389.783°

3.

   a. Arclength $= r\theta = \frac{4\pi}{3} = 4.189$ m; Area $= \frac{1}{2} r^2\theta = \frac{8\pi}{3} = 8.378$ m^2

   b. Arclength = 12.5 m; Area = 31.25 m^2

   c. $45° \equiv \frac{\pi}{4} = 0.7854^c \therefore$ Arclength = 7.854 m; Area = 39.27 m^2

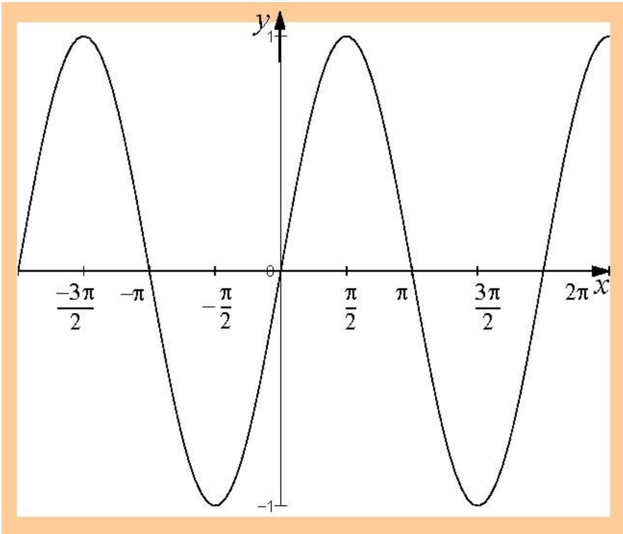
   d. $130° \equiv 2.269^c \therefore$ Arclength = 13.614 m; Area = 40.841 m^2

5.6 Graphs involving trigonometric functions

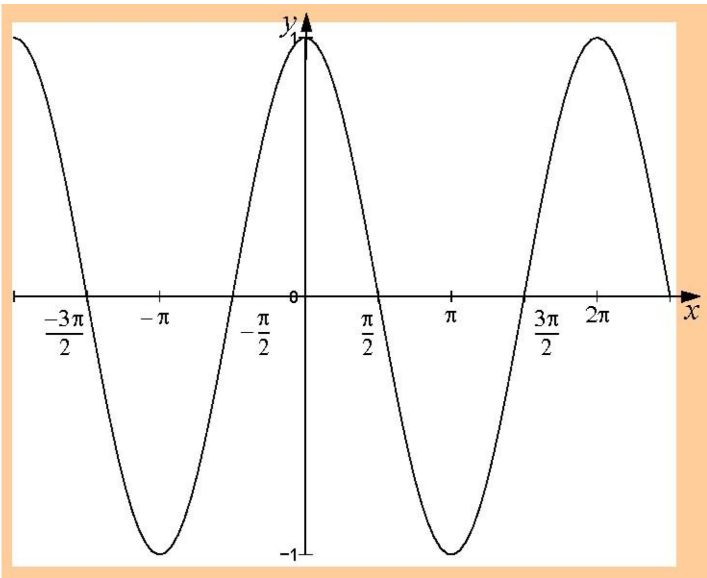
In order to use graphical methods to solve equations involving trig. functions, it is necessary to know the graphs of the basic trig. functions.

Using **radian** measure and your calculator, verify the following graphs of the trig. Function:

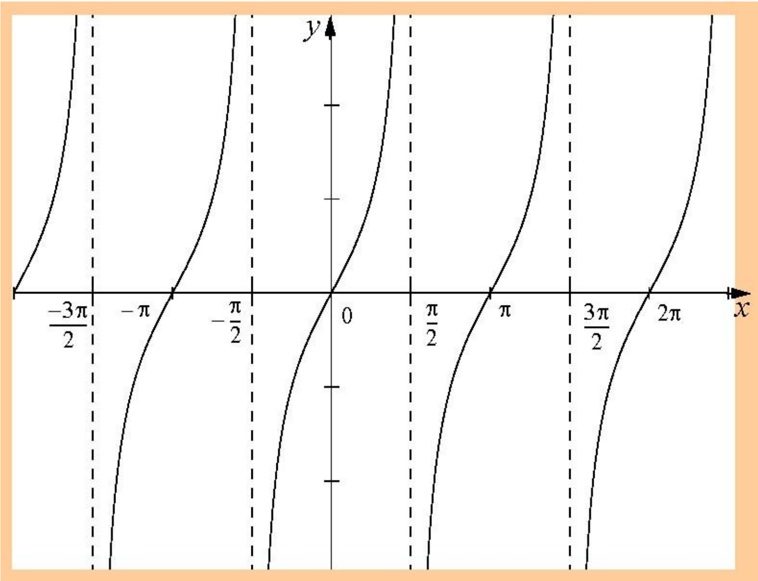
1. $y = \sin x$



2. $y = \cos x$



3. $y = \tan x$



Note that the ***x*-axis is measured in radians**. In most graphs involving trig functions, other functions of x are also involved e.g. $x - \sin x$. In such situations **radians must be used** since radians and trig functions are all ratios (i.e. they are numbers) and may be mixed. Degrees are not ratios and cannot be combined with trig functions.

Exercise 5.16

Sketch the graph of $y = \sec x$

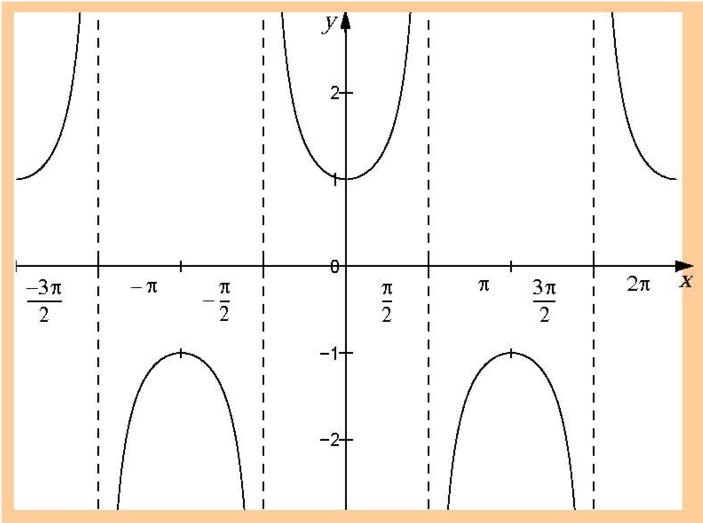
Solution

$$y = \sec x = \frac{1}{\cos x}$$

$\therefore$ Plot the inverse of the values for $\cos x$

x	$-\frac{\pi}{2}$	$-\frac{3\pi}{8}$	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\rightarrow$
$\cos x$	0	0.383	0.707	0.924	1	0.924	0.707	0.383	0	$\rightarrow$
$\sec x = 1/\cos x$	Undef	2.613	1.41	1.08	1	1.08	1.41	2.613	Undef	$\rightarrow$

$\rightarrow$	x	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π	$\frac{9\pi}{8}$	$\frac{5\pi}{4}$	$\frac{11\pi}{8}$	$\frac{3\pi}{2}$
$\rightarrow$	$\cos x$	–0.383	–0.707	–0.924	–1	–1.08	–0.707	–0.383	0
$\rightarrow$	$\sec x = 1/\cos x$	–2.613	–1.41	–1.08	–1	–0.924	–1.41	–2.613	Undef


y
2
1
0
-1
-2
x
$\frac{-3\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

Exercise 5.17

Find all solutions of $x^2 - 1 - \sin x = 0$

Solution

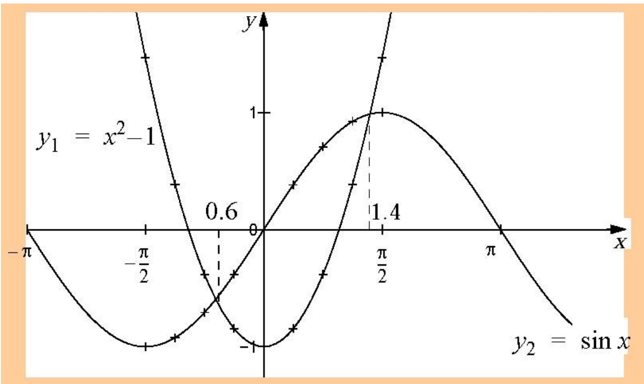
$y_1 = x^2 - 1, y_2 = \sin x$

The points of intersection of these graphs are the solutions.

Note: i. **Radians must** be used.

ii. we already know the general shapes of y_1 and y_2, (y_1 is a parabola and y_2 is the sine curve)

x	$-\frac{\pi}{2}$	$-\frac{3\pi}{8}$	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
$y_1 = x^2 - 1$	1.47	0.39	–0.38	–0.85	–1	–0.85	–0.38	0.39	1.47
$y_2 = \sin x$	−1	–0.92	–0.71	–0.38	0	0.38	0.71	0.92	1



There are two solutions at approximately $x \approx -0.6$ and $x \approx 1.4$. Note that although π/2 and π are marked on the x-axis, π is a very inconvenient constant to use for a scale, e.g. 10 divisions equal to 3.14159 radians is not easy to work with. It is better to pick a round number e.g. 10 divisions equals 1 radian.

Exercise 5.18

Use graphical methods to solve the equation $x - \tan x = 0$.

Solution

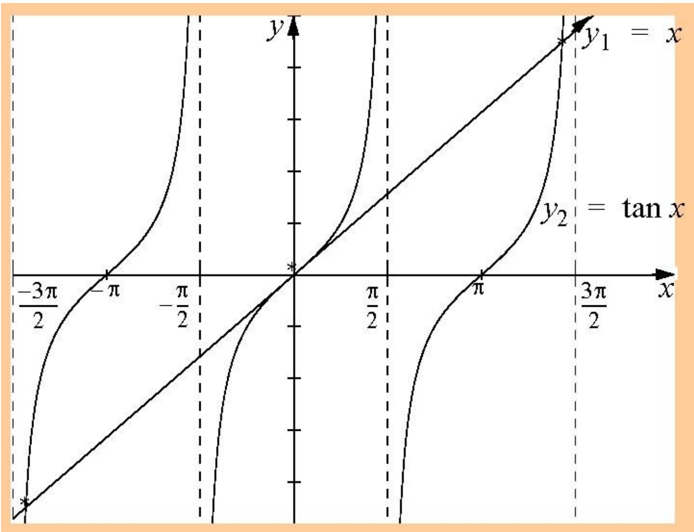
This can be written as $x = \tan x$.

Let $y_1 = x$ and $y_2 = \tan x$.

The points of intersection of these graphs are the solutions.

Note: i. **Radians must** be used.

ii. we already know the general shapes of y_1 and y_2, (y_1 is a straight line and y_2 is the tangent curve).



One solution is obviously $x = 0$ but other solutions exist at approximately

$$x = \pm\frac{3\pi}{2} \text{ (i.e. } x = \pm 4.5\text{)}, \quad x = \pm\frac{5\pi}{2} \text{ (i.e. } x = \pm 7.7\text{), etc.}$$

More accurate answers can be obtained by using more accurate graphs.

Additional exercises

Find all solutions of the following equations:

a. $x - \sin x = 0$

b. $x^2 - 1 - 2\sin x = 0$

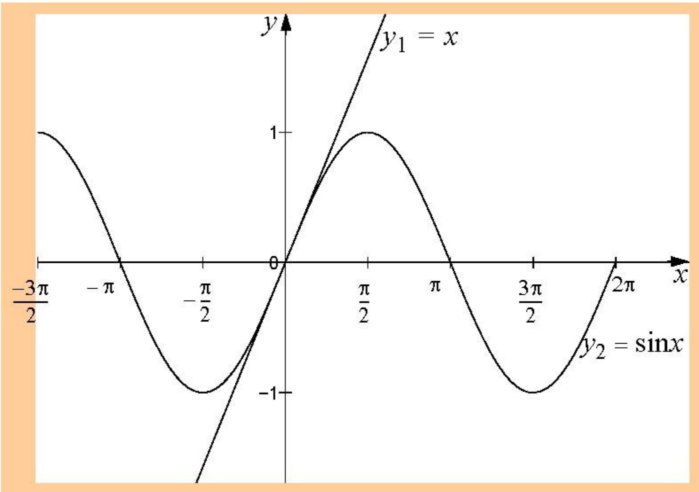
c. $x^2 - 3x + 4\cos x = 0$

Solutions

a. $x = \sin x \quad \therefore y_1 = x, \quad y_2 = \sin x$

(Note that **radians must be used** for all of these exercises since they all involve mixtures of trig and other functions of x.)

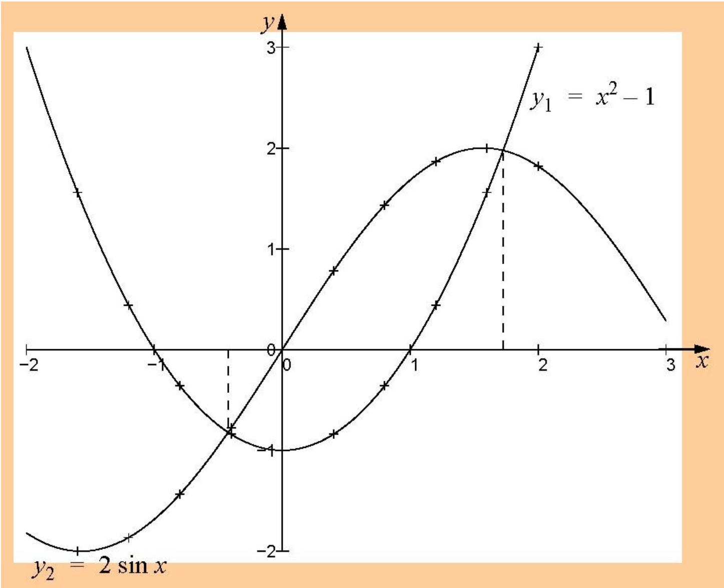
x	−1	−0.8	−0.6	−0.4	−0.2	0	0.2	0.4	0.6	0.8	1
y_1	−1	−0.8	−0.6	−0.4	−0.2	0	0.2	0.4	0.6	0.8	1
y_2	−0.84	−0.72	−0.56	−0.39	−0.199	0	0.199	0.39	0.56	0.72	0.84



The only solution is $x = 0$

b. $x^2 - 1 = 2\sin x \qquad \therefore y_1 = x^2 - 1, \qquad y_2 = 2\sin x$

x	−1.2	−0.8	−0.4	0	0.4	0.8	1.2	1.6	2
y_1	0.44	−0.36	−0.84	−1	−0.84	−0.36	0.44	1.56	3
y_2	−1.86	−1.43	−0.78	0	0.78	1.43	1.86	2.00	1.82



Solutions are roughly $x = -0.4$ and $x = 1.7$

c. $4\cos x = 3x - x^2 \qquad \therefore y_1 = 4\cos x, \qquad y_2 = 3x - x^2$

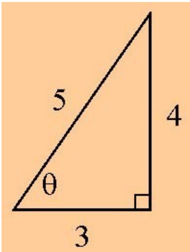
x	0	0.5	1	1.5	2	2.5	3	3.5	4
y_1	4	3.51	2.16	0.28	−1.66	−3.20	−3.96	−3.75	−2.61
y_2	0	1.25	2	2.25	2	1.25	0	−1.75	−4

Solutions are roughly $x = 1.05$ and $x = 3.80$

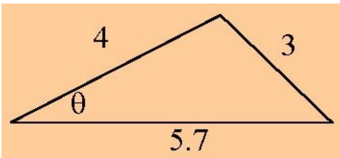
Module 5: Self assessment

Questions 5.1

1. In the triangle shown find $\sec\theta$.



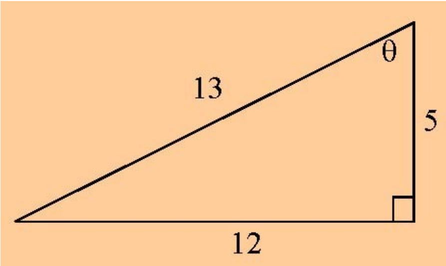
2. Find all the angles between 0 and 2π radians which satisfy $\sin\theta = 0.4$
3. Simplify $\sqrt{1 + \cot^2\theta}$
4. In the triangle given, find θ.



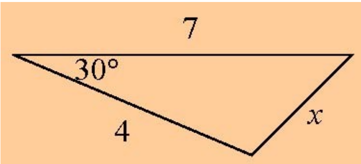
5. Find the area of the triangle in 4.
6. Simplify $\sin a \cos b + \cos a \sin b$
7. Convert an angle of $\frac{\pi}{6}$ radians to degrees.
8. If $\cos\theta = 0.3$, find $\cos 2\theta$
9. Sketch $y = \tan x$ for $-2\pi < x < 2\pi$

Questions 5.2

1. Convert $\frac{\pi}{3}$ radians into degrees.
2. In the triangle shown, find cosec θ



3. Simplify $\sqrt{1 - \cos^2 \theta}$
4. Solve the equation $\cos\theta = 0.4$ for $-2\pi < \theta < 2\pi$
5. In the triangle given find x.



6. Find the area of the triangle in 5.
7. Prove $\sin\theta \sec\theta \cot\theta = \sin^2\theta + \cos^2\theta$
8. If $\sin\theta = 0.8$, find $\sin 2\theta$
9. Sketch the graph of $y = \sin 2x$ for $-2\pi < x < 2\pi$